BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES
MATHÉMATIQUES, ASTRONOMIQUES ET PHYSIQUES

Volume IX Numéro 5

VARSOVIE 1961

Le Bulletin de l'Académie Polonaise des Sciences est publié en Séries.

La présente Série comprend des communications du domaine de la mathématique, astronomie et physique

Lesdites sciences sont du domaine de la Trosième Classe de l'Académie Polonaise des Sciences.

La présente Série paraît mensuellement.

Les communications destinées à être publiées dans le Bulletin sont présentées à la Rédaction du Bulletin par les membres l'Académie.

Adresse de la Rédaction: Pałac Kultury i Nauki, Varsovie

BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

Rédacteur en chef
K. KURATOWSKI

Rédacteur en chef suppléant
L. INFELD

SÉRIE DES SCIENCES
MATHÉMATIQUES, ASTRONOMIQUES ET PHYSIQUES

Rédacteur de la Série
A. MOSTOWSKI

Comité de Rédaction de la Série
K. BORSUK, W. IWANOWSKA, A. JABŁOŃSKI, W. RUBINOWICZ

VOLUME IX
NUMÉRO 5

VARSOVIE 1961

PRINTED IN POLAND

PAŃSTWOWE WYDAWNICTWO NAUKOWE, W-WA, MIODOWA 10

Nakład 1082+168 egz.	*Rękopis dostarczono 11.IV.1961*
Ark. wyd. 7,25, druk. 6,5+0,375+0,25	*Podpisano do druku 5.VI.61*
Papier druk. sat. 80 g kl. III	*Druk ukończono w czerwcu 1961*
Format B5, 70×100 cm	*Zam. prod. 159/61 Cena zł 20.—*

WARSZAWSKA DRUKARNIA NAUKOWA, W-WA, ŚNIADECKICH 8

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 5, 1961

MATHEMATICS

On Classes of Abelian Groups

by

S. BALCERZYK

Presented by K. BORSUK on March 19, 1961

1. The very important generalization of the classical Hurewicz theorem on homotopy and homology groups of topological space is due to J.P. Serre and is formulated in terms of classes of Abelian groups (see [2] or [1]).

A (not empty) collection $\mathcal{C}$ of Abelian groups is called a *class* if and only if the following conditions are satisfied:

(i) If a group A is isomorphic to some group in $\mathcal{C}$, then A is in $\mathcal{C}$.

(ii) If a group is a subgroup or a factor group of a group in $\mathcal{C}$, then A is in $\mathcal{C}$.

(iii) If an Abelian group A is an extension of a group in $\mathcal{C}$ by a group in $\mathcal{C}$, then A is in $\mathcal{C}$.

The groups belonging to a given class are considered in applications as "*little*" groups.

The following properties of classes are of great importance:

(I) If A and B are in $\mathcal{C}$, then $A \otimes B$ and Tor (A, B) are in $\mathcal{C}$.

(II) If A is in $\mathcal{C}$, then $H_n(A)$ is in $\mathcal{C}$ for all $n > 0$. *)

A class with property (I) is called *weakly complete*, that with property (II) — perfect. The author knows of no class without property (I) or (II) (see [2], [1] p. 309) and the purpose of this note is to give an example of such a class.

Our subsequent paper will be devoted to description of some new classes and to more exact study of sufficient conditions for a class to be weakly complete. E.g., the following theorem holds: $\mathcal{C}$ is weakly complete, if it has the property that together with the p-primary countable groups containing A_p it also contains their direct sum $\sum\limits_p A_p$.

2. We shall consider only torsion groups. Every single group A may be represented uniquely as the direct sum of its primary components: $A = \sum\limits_{k=1}^{\infty} A_k$, A_k being p_k-primary and $p_1, p_2, \ldots$ — a sequence of all primes. If a p_k-primary group A_k admits a finite number of generators, then dim A_k is the cardinality of a minimal

*) $H_n(A)$ is n-th homology group (with integer coefficients) of the group A.

set of generators. We shall denote by C_m a cyclic group of the order m and by C_m^r the direct sum of r copies of the group C_m.

The following properties of functors $\otimes$ and Tor are well known:

$$C_m \otimes A \approx A/mA, \tag{2.1}$$

$$\operatorname{Tor}(C_m, A) = \{a \in A,\ ma = 0\}, \tag{2.2}$$

$$H_n(C_m) = \begin{cases} C_m \text{ for odd } n \\ 0 \text{ for even, positive } n. \end{cases} \tag{2.3}$$

Using Künneth relations one can express homology groups of direct sum $A+B$ by homology groups of A and B:

$$H_n(A+B) = \sum_{i=0}^{n} H_i(A) \otimes H_{n-i}(B) + \sum_{i=1}^{n-2} \operatorname{Tor}(H_i(A), H_{n-i-1}(B)). \tag{2.4}$$

(2.5) If A is a torsion group with p_k-primary components A_k: $A = \sum_{k=1}^{\infty} A_k$ then for $n > 0$ $H_n(A_k)$ is p_k-primary and $H_n(A) = \sum_{k=1}^{\infty} H_n(A_k)$.

3. Let $\mathcal{C}_0$ be a class consisting of all torsion groups A with finite primary components A_k and satisfying condition:

$$\textit{the sequence } \left\{\frac{1}{k}\dim A_k\right\} \textit{ is bounded.} \tag{3.1}$$

It is easy to verify that all conditions (i)—(iii) hold.

Let us define the group $B = \sum_{k=1}^{\infty} B_k$ with $B_k = C_{p_k}^k$. We shall prove the following:

THEOREM. *The group B is in the class $\mathcal{C}_0$ and none of the groups $B \otimes B$,* Tor (B, B), *$H_n(B)$ $(n \geqslant 2)$ belongs to $\mathcal{C}_0$.*

First we shall prove two lemmas.

LEMMA 1. *If $B' = B \otimes B = \sum_{k=1}^{\infty} B'_k$, $B'' = \operatorname{Tor}(B, B) = \sum_{k=1}^{\infty} B''_k$ then* $\dim B'_k = \dim B'' = k^2$.

Proof. By the additivity of functors $\otimes$ and Tor we have $B' = \sum_{n,k=1}^{\infty} B_n \otimes B_k$, $B'' = \sum_{n,k=1}^{\infty} \operatorname{Tor}(B_n, B_k)$. It is known that $B_n \otimes B_k = 0 = \operatorname{Tor}(B_k, B_k)$ for $n \neq k$; consequently, $B'_k = B_k \otimes B_k$ and $B'' = \operatorname{Tor}(B_k, B_k)$. By (2.1) and (2.2) $C_p \otimes C_p \approx \approx C_p \approx \operatorname{Tor}(C_p, C_p)$ for each prime p. The Lemma follows by the additivity of both functors.

LEMMA 2. *For each prime p, $n \geqslant 1$ the group $H_n(C_p^r)$ is finite, consists of elements of order p and* $\dim H_n(C_p^r) \geqslant \dim H_2(C_p^r) = \dfrac{r(r-1)}{2}$ *for $n \geqslant 2$, $r \geqslant 2$.*

Proof. The first part of the Lemma follows inductively (with respect to r) from (2.4). To prove the second part we shall give a recursive formula for $\dim H_n(C_p^r)$.

Let us denote $s(n, r) = \dim H_n(C_p^r)$, then $H_n(C_p^r) = C_p^{s(n,r)}$. Since $C_p^{r+1} = C_p + C_p^r$ then, by (2.4) and (2.1), (2.2), we have

$$H_n(C_p^{r+1}) = \sum_{i=0}^{n} H_i(C_p) \otimes H_{n-i}(C_p^r) + \sum_{i=1}^{n-2} \operatorname{Tor}(H_i(C_p), H_{n-1-i}(C_p^r)) =$$

$$= C_p^{s(n,r)} + H_n(C_p) + \sum_{i=1}^{n-1}{}' C_p^{s(n-i,r)} + \sum_{i=1}^{n-2}{}' C_p^{s(n-1-i,r)}$$

(in the sums $\sum'$ all indexes i are odd).

If n is even and positive then we get the recursive formula $s(n, r+1) = s(n, r) + \sum_{i=1}^{n-1} s(n-i, r) + \sum_{i=1}^{n-2} s(n-1-i, r) = \sum_{i=1}^{n} s(i, r)$; if n is odd then $s(n, r+1) = s(n, r) + 1 + \sum_{i=1}^{n-1} s(n-i, r) + \sum_{i=1}^{n-2} s(n-1-i, r) = 1 + \sum_{i=1}^{n} s(i, r)$. Finally

$$(3.2) \qquad s(n, r+1) = \begin{cases} s(i, r) \text{ for even, positive } n, \\ 1 + \sum_{i=1}^{n} s(i, r) \text{ for odd } n. \end{cases}$$

It is known, that $H_1(A) = A$, hence $s(1, r) = r$; moreover $s(2, r+1) = s(1, r) + s(2, r) = r + s(2, r)$ and $s(2, 1) = 0$ imply $s(2, r) = \frac{r(r-1)}{2}$. Formula (3.2) implies $s(n, r) \geqslant s(1, r-1) = r-1 > 0$ for $n \geqslant 1$ and $r \geqslant 2$, hence using (3.2) once more we get for $n \geqslant 2$, $r \geqslant 2$ $s(n+1, r) \geqslant \sum_{i=1}^{n=1} s(i, r-1) \geqslant 1 + \sum_{i=1}^{n} s(i, r-1) \geqslant s(n, r)$ and the proof is completed.

Proof of the theorem. The first part follows from Lemma 1, because $1/k \dim B_k' = 1/k \dim B_k'' = k$; moreover, $1/k \dim B_k = 1$. By (2.5) we have for $n > 0$ $H_n(B) = \sum_{k=1}^{\infty} H_n(B_k)$, $H_n(B_k)$ being primary. Since $B_k = C_{p_k}^k$, then by Lemma 2, $1/k \dim H_n(B_k) \geqslant \frac{k-1}{2}$ for $n \geqslant 2$ and $k \geqslant 2$; thereby the groups $H_n(B)$ (for $n \geqslant 2$) do not belong to $\mathcal{C}_0$.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY, PAN)

REFERENCES

[1] S. T. Hu, *Homotopy Theory*, New York, 1959.

[2] J. P. Serre, *Groupes d'homotopie et classes de groupes abeliens*, Ann. of Math., **58** (1953), 258.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 5, 1961

MATHEMATICS

Negative Solution of I. Kaplansky's First Test Problem for Abelian Groups and a Problem of K. Borsuk Concerning Cohomology Groups

by

E. SĄSIADA

Presented by A. MOSTOWSKI on February 14, 1961

I. Kaplansky's first test problem for Abelian groups is to establish the following question [5].

Are two Abelian groups isomorphic, if each one is isomorphic to a direct summand of the other?

The purpose of this note is to show that the answer to this problem is negative.

We shall prove that the pair of torsion free Abelian groups X and Y defined in the subsequent section, satisfy the following two conditions:

(i) X is isomorphic to a direct summand of Y and Y is isomorphic to a direct summand of X.

(ii) X is not isomorphic to Y.

From the announced result and a theorem of C. T. Yung [7] a negative answer to the following K. Borsuk's problem about cohomology groups [2] may be deduced.

Have two topological spaces isomorphic cohomology groups, if these are R-equivalent (i.e. each one is homeomorphic to a retract of the other)?

Construction of the groups X and Y

We denote by $R^{(p)}$ (p is a prime) the additive group of all rationals k/p^n with integers k and n.

Let C be the subgroup of $R^{(5)}+R^{(3)}$ generated by $2\,R^{(5)}+2\,R^{(3)}$ and an element $a-c$ where $a \in R^{(5)} \setminus 2\,R^{(5)}$, $c \in E^{(3)} \setminus 2\,R^{(3)}$.

Let D be the subgroup of $R^{(5)}+R^{(2)}$ generated by $3\,R^{(5)}+3\,R^{(2)}$ and an element $b-d$ where $b \in R^{(5)} \setminus 3\,R^{(5)}$, $d \in R^{(2)} \setminus 3\,R^{(2)}$.

We define the groups X and Y as follows:

$$X = \sum_{i=1}^{\infty}{}^{*} C_i + \sum_{i=1}^{\infty} D_i,$$

where $\sum_{i=1}^{\infty}{}^{*} C_i$ is the complete direct sum of groups $C_i = C\,(i = 1, 2, \ldots)$ and $\sum_{i=1}^{\infty} D_i$ is the discrete direct sum of groups $D_i = D\,(i = 1, 2, \ldots)$

$$Y = X + A, \text{ where } A = R^{(5)}.$$

Proof of (i) and (ii)

First of all we establish the following facts.

(1) Both the groups C and D are indecomposable [4].

(2) The group $C + D$ has a direct summand isomorphic to $R^{(5)}$ [4].

(3) If U is a directly indecomposable direct summand of $\sum_{i=1}^{n} C_i$, where $C_i = C\,(i = 1, 2, \ldots, n)$, then U is isomorphic to C.

Proof. At first observe that

(3.1) if U is an indecomposable subgroup of $G = \sum_{k=1}^{n_1} R_k^{(5)} + \sum_{k=1}^{n_2} R_k^{(3)}$ $(n_1, n_2 \geqslant 1)$ containing $2\,G$, then $n_1 = n_2 = 1$ and therefore U is isomorphic to C.

In fact, let $\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_l$ be the basis of $U/2\,G$. Then $U = \{2\,G, c_1, c_2, \ldots, c_l\}$. Consider the decompositions $c_i = a_i + b_i$ where $a_i \in \sum_{k=1}^{n_1} R_k^{(5)}$ and $b_i \in \sum_{k=1}^{n_2} R_k^{(3)}$. Since c_i are linearly independent, then there exists a decomposition $G = \bar{R}_1^{(5)} + \bar{R}_1^{(3)} + G_1$ where $a_1 \in \bar{R}_1^{(5)}$, $b_1 \in \bar{R}_1^{(3)}$ and $c_i \in G_1$ $(i = 2, \ldots, l)$. Let U_1 be the subgroup of $\bar{R}_1^{(5)} + \bar{R}_1^{(3)}$ generated by $2\,\bar{R}_1^{(5)} + 2\,\bar{R}_1^{(3)}$ and element c_1. Evidently, we have $U_1 \subset U$ and $U \subset U_1 + G_1$. From here in view of the assumption that U is indecomposable we obtain $U = U_1$ which completes the proof of (3.1).

Let now $\sum_{i=1}^{n} C_i = U + W$. The group $\sum_{i=1}^{n} C_i$ can be represented as the subgroup of $\sum_{i=1}^{n} R_i^{(5)} + \sum_{i=1}^{n} R_i^{(3)}$ generated by $\sum_{i=1}^{n} 2\,R_i^{(5)} + \sum_{t=1}^{n} 2\,R_i^{(3)}$ and the elements $a_i - c_i$ $(i = 1, 2, \ldots, n)$ where $a_i \in R_i^{(5)} \setminus 2\,R_i^{(5)}$ and $c_i \in R_j^{(3)} \setminus 2\,R_i^{(3)}$.

Since the subgroups $\sum_{i=1}^{n} 2\,R_i^{(5)}$ and $\sum_{i=1}^{n} 2\,R_i^{(3)}$ are fully characteristic in $U + W$

$$\sum_{i=1}^{n} 2\,R_i^{(5)} + \sum_{i=1}^{n} 2\,R_i^{(3)} = U \cap \sum_{i=1}^{n} 2\,R_i^{(5)} + U \cap \sum_{i=1}^{n} 2\,R_i^{(3)} + W \cap \sum_{i=1}^{n} 2\,R_i^{(5)} + W \cap \sum_{j=1}^{k} 2\,R_j^{(0)}. \tag{3.2}$$

We show that

$$\text{if } U \neq \{0\}, \text{ then } U \cap \sum_{i=1}^{n} 2R_i^{(5)} \neq \{0\} \neq U \cap \sum_{i=1}^{n} 2R_i^{(3)}. \tag{3.3}$$

Suppose that $U \cap \sum_{i=1}^{n} 2R_i^{(5)} = \{0\}$. Then $\sum_{i=1}^{n} 2R_i^{(5)} \subset W$. Let $2c_i = 2\,c_i' + 2c_i''$ where $2c_i' \in U$, $2c_i'' \in W$. Then $a_i - c_i = -c_i' + (a_i - c_i'')$ where $c_i' \in U$ and

$a_i - c_i'' \in W$. Since $c_i' \in \sum_{i=1}^{n} C_i \cap \sum_{i=1}^{n} R_i^{(3)} = \sum_{i=1}^{n} 2R_i^{(3)}$, then $c_i' = 2c_i$ $(i = 1, 2, \ldots, n)$. Further observe that the elements c_i'' $(i = 1, 2, \ldots, n)$ are linearly independent. (In fact: if $\sum_{i=1}^{n} k_i c_i'' = 0$ with $(k_1, \ldots, k_n) = 1$, then $\sum_{i=1}^{n} k_i c_i = \sum_{i=1}^{n} k_i c_i' = 2\sum_{i=1}^{n} k_i c_i$. This and the assumption c_i non $\in 2R_i^{(3)}$ imply that 2 divides each k_i which contradicts $(k_1, \ldots, k_n) = 1$). Since $2c_i'' \in W$ $(i = 1, 2, \ldots, n)$ and c_i'' $(i = 1, 2, \ldots, n)$ are linearly independent, then $\sum_{i=1}^{n} 2R_i^{(3)} \subset W$. Consequently $\sum_{i=1}^{n} C_i \subset W$, thus $U = \{0\}$.

The summands of the right hand side of Eq. (3.2) are direct summands of fully decomposable groups $\sum_{i=1}^{n} 2R_i^{(5)}$ and $\sum_{i=1}^{n} 2R_i^{(3)}$, respectively, therefore they are fully decomposable themselves [1].

Let

$$(3.4) \qquad U \cap \sum_{i=1}^{n} 2R_i^{(5)} = \sum_{k=1}^{n} 2R_k^{(5)}, \qquad U \cap \sum_{i=1}^{n} 2R_i^{(3)} = \sum_{k=1}^{n} 2R_k^{(3)}$$

From (3.3.) and (3.4.) follows

$$(3.5) \qquad U \subset \sum_{k=1}^{n_1} R_k^{(5)} + \sum_{k=1}^{n_2} R_k^{(3)} \text{ and } \sum_{k=1}^{n_1} 2R_k^{(5)} + \sum_{k=1}^{n_2} 2R_k^{(3)} \subset U \qquad (n_2, n_2 \geq 1)$$

Thus by (3.5.) and (3.1.) we obtain finally that U is isomorphic to C.

An argument similar to that used in the proof of (3) proves

(3′) If V is a directly indecomposable direct summand of $\sum_{i=1}^{n} D_i$ where $D_i = D$ $(i = 1, 2, \ldots, u)$, then V is isomorphic to D.

(4) Every countable and reduced torsion free Abelian group is slender [6].

(5) Every homomorphism of a complete direct sum $\sum_{i=1}^{n}{}^{*} B_i$ into a slender group sends almost all components B_i onto zero (Theorem of J. Łoś).*)

From (3) and (5) follows

(6) Every countable direct summand K of $\tilde{\sum_{i=1}^{}}{}^{*} C_i$ $(C_i = C$ $i = 1, 2, \ldots)$ is of form $K = \sum_{i=1}^{k} C_i''$ where $C_i'' \simeq C$ $(i = 1, 2, \ldots, k)$.

From (3′) follows

(7) Every direct summand L of finite rank of the discrete direct sum $\tilde{\sum_{i=1}^{}} D_i$ $(D_i = D$ $i = 1, 2, \ldots)$ has the form $L = \sum_{i=1}^{l} D_i''$, where $D_i'' \simeq D$ $(i = 1, 2, \ldots, l)$.

We approach now to the proof of (1) and (ii).

It follows immediately from the definition of groups X and Y and of (2) that the groups X and Y satisfy condition (i).

*) For proof see [3].

To prove (ii) we suppose that X is isomorphic to Y. Then we have

$$\sum_{i=1}^{\infty}{}^{*} C_i + \sum_{i=1}^{\infty} D_i = \sum_{i=1}^{\infty}{}^{*} C_i' + \sum_{i=1}^{\infty} D_i' + A, \tag{9}$$

where $C_i \simeq C_i' \simeq C$, $D_i \simeq D_i' \simeq D$, $A \simeq R^{(5)}$.

Since by (4) the group $\sum_{i=1}^{\infty} D_i' + A$ is slender (5) shows that the projection of $\sum_{i=1}^{\infty}{}^{*} C_i$ into $\sum_{i=1}^{\infty} D_i' + A$ sends almost all components C_i onto zero. Thus there exists a number n such that $\sum_{i=n+1}^{\infty}{}^{*} C_i \subset \sum_{i=1}^{\infty}{}^{*} C_i'$. We have $\sum_{i=1}^{\infty}{}^{*} C_i' = \sum_{i=nH}^{\infty}{}^{*} C_i + K$, where K as a group isomorphic to a subgroup of $\sum_{i=1}^{\infty} C_i + \sum_{i=1}^{\infty} D_i$ is countable. Hence, by (6) we have

$$K = \sum_{i=1}^{k} C_i'' \qquad (C_i'' \simeq C, (i = 1, 2, \ldots, k)).$$

Hence, by (9), we obtain

$$\sum_{i=1}^{n} C_i + \sum_{i=1}^{\infty} D_i \simeq \sum_{i=1}^{k} C_i'' + \sum_{i=1}^{\infty} D_i' + A. \tag{10}$$

It follows from (10) that $\sum_{i=1}^{k} C_i'' + A \subset \sum_{i=1}^{n} C_i + \sum_{i=1}^{m} D_i$ for a suitable m and

$$\sum_{i=1}^{n} C_i + \sum_{i=1}^{m} D_i = \sum_{i=1}^{k} C_i'' + A + L. \tag{11}$$

From (11), (10) and (7) it follows that $L = \sum_{i=1}^{l} D_i''$ where $D_i'' \simeq D$ $(i = 1, 2, \ldots, l)$.

Thus from the assumption $X \simeq Y$ we obtain the following equation

$$\sum_{i=1}^{n} C_i + \sum_{i=1}^{m} D_i = \sum_{i=1}^{k} C_i'' + \sum_{i=1}^{l} D_i'' + A \quad (= M) \tag{12}$$

Let M_p be the maximal subgroup of M consisting of all elements divisible by each power p^n $(n = 1, 2, \ldots)$.

It is easy to see that for the left hand side of Eq. (12) $r(M_2) = m$, $r(M_3) = n$, $r(M_5) = n + m$ and for the right hand side of this equation $r(M_2) = l$, $r(M_3) = k$ and $r(M_5) = l+k+1$. ($r(M_p)$ rank of M_p).

The obtained contradiction $k + l = k + l + 1$ shows that X is not isomorphic to Y.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY, PAN)

REFERENCES

[1] R. Baer, *Abelian groups without elements of finite order*, Duke Math. J. 3 (1937), 68—122.

[2] S. Eilenberg, *On problems of topology*, Ann. of Math. **50** (1949), 247—260.

[3] L. Fuchs, *Abelian groups*, Budapest, 1958.

[4] B. Jónsson, *On direct decompositions of torsion free Abelian groups*, Math. Scand. 5 (1957), 230—235.

[5] I. Kaplansky, *Infinite Abelian groups*, Ann. Arbor, 1954.

[6] E. Sąsiada, *Proof that every countable and reduced torsion free Abelian group is slender*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys. **7** (1959), 147—148.

[7] C. T. Yung, *On Borsuk's problem*, Summa Brasil. Math. **3** (1952), 13—20.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 5, 1961

MATHEMATICS

Some Remarks on the Class $\mathcal{L}$ of Probability Distributions

by

L. KUBIK

Presented by E. MARCZEWSKI on March 1, 1961

Let us consider the sequence

$$\xi_{n1}+\xi_{n2}+\dots+\xi_{nk_n}-A_n \quad (n=1,2,\dots), \tag{1}$$

where $A_n=$ const and the random variables ξ_{nk} $(k=1,2,\dots,k_n)$ are independent and uniformly asymptotically negligible, i.e. for every $\varepsilon>0$

$$\lim_{n\to\infty}\max_{1\leqslant k\leqslant k_n} P(|\xi_{nk}|>\varepsilon)=0.$$

It is known that the class of all possible limiting distributions of sums (1) is equal to the class of all infinitely divisible distributions. The class of infinitely divisible distributions can be characterized as follows (see Gnedenko and Kolmogorov [2], § 17, theorem 5):

(i) The class of infinitely divisible distributions is equal to the class of compositions of finite number of Poisson distributions and of their limits (in the sense of weak convergence).

Let us now consider the cumulative sums of independent random variables

$$\frac{\xi_1+\xi_2+\dots+\xi_n}{B_n}-A_n, \qquad A_n, B_n=\text{const}, \qquad B_n>0, \tag{2}$$

where the random variables ξ_k/B_n $(k=1,2,\dots,n)$ are uniformly asymptotically negligible. The class of limiting distributions of sums (2) is called the class $\mathcal{L}$. The aim of this note is to give such a characterization of the class $\mathcal{L}$ which would correspond to characterization (i) of the class of infinitely divisible distributions.

If X has the distribution from the class $\mathcal{L}$, then the logarithm of the characteristic function $\varphi(t)$ of X can be written in the Lévy—Khintchine's form

$$\log\varphi(t)=i\gamma t+\int_{-\infty}^{+\infty}\left(e^{itu}-1-\frac{itu}{1+u^2}\right)\frac{1+u^2}{u^2}\,dG(u). \tag{3}$$

γ is a real number and $G(u)$ is a non-decreasing bounded function ($G(-\infty)=0$) having the right derivative and the left derivative, denoted indifferently by $G'(u)$,

at every point $u \neq 0$ and such that

$$\frac{1+u^2}{u} G'(u)$$

is for $u < 0$ and for $u > 0$ non-increasing function (Gnedenko and Groshev [1], p. 522).

Let us consider the distributions with function $G(u)$ of the form

$$G(u) = \begin{cases} 0 \quad \text{for} \quad u < A, \\ a \log \dfrac{1+A^2}{1+u^2} \quad \text{for}\, A \leqslant u \leqslant 0, \\ a \log(1+A^2) \quad \text{for}\, u > 0 \end{cases} \quad (a \geqslant 0) \tag{4}$$

or

$$G(u) = \begin{cases} 0 \text{ for } u \leqslant 0, \\ b \log(1+u^2) \text{ for } 0 < u \leqslant B, \\ b \log(1+B^2) \text{ for } u > B, \end{cases} \quad (b \geqslant 0) \tag{5}$$

or

$$G(u) = \begin{cases} 0 \text{ for } u \leqslant 0, \\ c \text{ for } u > 0. \end{cases} \quad (c \geqslant 0) \tag{6}$$

Let us denote by $\mathcal{G}$ the class of all such distributions. The distribution from the class $\mathcal{G}$ is, for the class $\mathcal{L}$, an analogon of the Poisson distribution for the class of infinitely divisible distributions, namely the following theorem holds:

THEOREM. *The class $\mathcal{L}$ of distributions is equal to the class of compositions of finite number of distributions from the class $\mathcal{G}$ and of their limits (in the sense of weak convergence).*

The proof of this theorem will appear in paper [4].

Let us observe yet that the analogy between the Poisson distribution and the distribution from the class $\mathcal{G}$ goes further. It is known that the Poisson distribution is the limiting distribution of sums (1), where ξ_{nk} are suitable chosen two-valued random variables. Similarly, every distribution from the class $\mathcal{G}$ is the limiting distribution of sums (2), where ξ_k are suitable chosen two-valued random variables. This follows immediately from paper [3] and from the relation between $G(u)$ and $K(u)$, where $G(u)$ is the function in Lévy-Khintchine's formula and $K(u)$ is the function in Kolmogorov's formula.

DEPARTMENT OF MATHEMATICAL STATISTICS, UNIVERSITY, WARSAW,
(KATEDRA STATYSTYKI MATEMATYCZNEJ, UNIWERSYTET WARSZAWSKI)

REFERENCES

[1] B. V. Gnedenko and A. V. Groshev, *On the convergence of laws of distributions of standarized sums of independent random variables*, [in Russian], Mat. Sb., **6** (1939), 521—541.

[2] B. V. Gnedenko and A. N. Kolmogorov, *Limit distributions for sums of independent random variables*, Moscow, 1949, [in Russian].

[3] L. Kubik, *The limiting distributions of cumulative sums of independent two-valued random variables*, Studia Math., **18** (1959), 296—309; **19** (1960), 249.

[4] — , *A characterization of the class L of probability distributions* (presented for publication in Studia Mathematica).

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 5, 1961

MATHEMATICS

Separable Groups. I

by

M. KRÓL

Presented by A. MOSTOWSKI on March 8, 1961

A torsion-free Abelian group G is said to be *separable*, if every finite subset of elements of G is contained in a completely decomposable direct summand of finite rank of G.

In the present paper we give some conditions for the separability of the complete direct sums of torsion-free Abelian groups of rank one. It is a complete solution of the problem of L. Fuchs ([1], Problem 25). Theorem 5.2 of the present paper was announced in [6] but condition (b′) was neglected.*)

We characterize the separable complete direct sums of torsion-free Abelian groups of rank one by their invariants, i.e., by the types of their components and powers of sets of components of the same type.

1. Notations and lemmas

All groups are additively written torsion-free Abelian groups. R, R_t denote groups of rank one. $\{R_t\}_{t\epsilon T}$ being a family of groups of rank one, we denote by $\sum\limits_{t\epsilon T}^{*} R_t$ the complete direct sum of the groups R_t.

If $g \neq 0$ is an element of Abelian torsion-free group G, $\chi(g)$ and $\tau(g)$ will denote, respectively, the characteristic and the type of g in the group G.

By a *homogeneous group* we mean a group in which all elements different from 0 are of the same type.

It follows from a theorem of R. Baer [2] that every torsion-free Abelian group of rank one is a homogeneous group. By $\tau(R_t)$ we denote the type of the group R_t of rank one.

Let $G = \sum\limits_{t\epsilon T}^{*} R_t$ be the complete direct sum of groups R_t $(t \in T)$ of rank one, $\Omega(G)$ — the set of all different types $\tau(R_t)$ of components R_t $(t \in T)$, and $T_{\mathfrak{A}}$ $(\mathfrak{A} \in \Omega(G))$ — the subset of T such that $\tau(R_t) = \mathfrak{A}$ for $t \in T_{\mathfrak{A}}$. With these notations assumed we mean by the canonical decomposition of group G the decom-

*) E. Sąsiada has informed me, that in the meantime Problem 25 was solved (independently) by A. P. Mishina and A. L. S. Corner, but their results have not been published as yet.

position $\sum\limits_{\mathfrak{A}\in\Omega(G)}^{*} G^{(\mathfrak{A})}$, where $G^{(\mathfrak{A})} = \sum\limits_{t\in T_{\mathfrak{A}}}^{*} R_t$. The sets $\Omega(G)$ and $T_{\mathfrak{A}}$ for $\mathfrak{A}\in\Omega(G)$ are invariants of the group G.

Let $\Omega_{(0\,\infty)}(G)$ be the subset of $\Omega(G)$ consisting of all types determined by characteristics with elements 0 and ∞ only, and let $\Omega_{*}(G) = \Omega(G)\setminus\Omega_{(0,\infty)}(G)$.

(1. 1) (R. Baer [2]). *If $G = \sum\limits_{t\in T}^{*} R_t$ is a complete direct sum of groups of rank one of the same type $\mathfrak{A}$, then the subgroup $G_{\mathfrak{A}} = \{x\in G;\ \tau(x)\geqslant\mathfrak{A}\}$ is separable.*

(1. 2) *If D is the maximal divisible subgroup of the group G and $G = D + G'$ then G is separable if and only if G' is separable.*

We call a torsion-free Abelian group L *slender*, if every homomorphism of a complete direct sum of countable set of infinite cyclic groups $\{a_n\}$ $(n = 1, 2, \ldots)$ into L sends almost all component $\{a_n\}$ into the zero of L.

2. The complete direct sums of isomorphic groups of rank one

First we prove the following

THEOREM 2.1. *The complete direct sum $G = \sum\limits_{t\in T}^{*} R_t$ of isomorphic groups R_t $(t\in T)$ of rank one is separable if and only if G is homogeneous.*

Proof. By (1. 1) our condition is sufficient. Let us prove that this condition is also necessary. Let $G = \sum\limits_{t\in T}^{*} R_t$, where $\tau(R_t) = \mathfrak{A}$ for all $t\in T$, and let G be separable. Then for every element $g\in G$, $g\neq 0$ there exists a direct summand H of finite rank which is completely decomposable and $g\in H$. Let $H = H_1 + \ldots + H_m$ be a decomposition of H into a direct sum of groups $H_1, \ldots, H_m$ of rank one. By results of A. P. Mishina [4] and E. Sąsiada [5] we have $\tau(H_j) = \mathfrak{A}$ for $j = 1, \ldots, m$, consequently, $\tau(g) = \mathfrak{A}$ for every $g\in G$ and G is a homogeneous group.

Now we shall give a characterization of complete direct sums of groups of rank one, which are homogeneous.

THEOREM 2.2. *Let $G = \sum\limits_{t\in T}^{*} R_t$ be a complete direct sum of isomorphic groups R_t, $\tau(R_t) = \mathfrak{A}$ $(t\in T)$. On this assumption we have:*

(a) *if $\mathfrak{A}\in\Omega_{(0,\infty)}(G)$, then G is homogeneous;*

(b) *if $\mathfrak{A}\in\Omega_{*}(G)$, then G is homogeneous if and only if the set T is finite.*

Proof. The condition (a) and sufficiency of the condition (b) is obvious. We shall prove, that if $\mathfrak{A}\in\Omega_{*}(G)$ and if G is a homogeneous group, then T is a finite set. If $\mathfrak{A}\in\Omega_{*}(G)$, then there exists an infinite sequence of positive integers $a_1 < a_2 < \ldots$ and a sequence of characteristics $\chi_m = (l_1^{(m)}, l_2^{(m)}, \ldots)$ $(m = 1, 2, \ldots)$ belonging to $\mathfrak{A}$ such that $0\neq l_{a_n}^{(m)} < \infty$, if $n\neq m$, and $l_{a_n}^{(m)} = 0$, if $n = m$ $(n = 1, 2, \ldots)$. Let us assume that the set T is of cardinality $\geqslant\aleph_0$. Then there exists an infinite sequence $(t_m)_{m=1,2,\ldots}$ of different elements of T and it is easy to show that the type of the element $g = (g_t)_{t\in T}$ such that $\chi(g_{t_m}) = \chi_m$ $(m = 1, 2, \ldots)$ and $g_t = 0$ for $t\neq t_m$ $(m = 1, 2, \ldots)$ is less than $\mathfrak{A}$. The manifest contradiction with the assumed homogeneity of G proves that the set T must be finite.

By the two previous theorems we have immediately:

COROLLARY 2.3. *Let $G = \sum^*_{t\in T} R_t$ be a complete direct sum of isomorphic groups R_t of rank one and type $\mathfrak{A}$. Then we have:*

(a) *if $\mathfrak{A} \in \Omega_{(0,\infty)}(G)$, G is separable and there are no restrictions onto the power of T;*

(b) *if $\mathfrak{A} \in \Omega^*(G)$, G is separable if and only if the set T is finite.*

3. Reduction of the problem

THEOREM 3.1. *If the group $G = \sum^*_{t\in T} R_t$ is separable and T' is a subset of T with cardinality of measure zero, then the group $G_1 = \sum^*_{t\in T'} R_t$ is separable, too.*)*

Proof. Let $g_1, \dots, g_n$ be a finite set of elements of G_1. The group G is separable. Then there exists a completely decomposable direct summand A of G of finite rank $(G = A+B)$ such that $g_1, \dots, g_n \in A$. We may assume G to be a reduced group (cf. (1, 2)) and then, by a theorem of E. Sąsiada [3], the group A is a slender group. Let h be the projection of G onto A. Then $h(G_1) \subset A$ and, by a theorem of J. Łoś ([1], th. 47,2), it follows that there exists a finite set $T_0' \subset T'$ such that the projection h sends the group $B_1 = \sum^*_{t\in T'\setminus T_0'} R_t$ into $\{0\}$. Since B_1 is a direct summand of G_1 and $h(B_1) = \{0\}$, i.e. $B_1 \subset B$, we have that B_1 is a direct summand of B. Let $B = B_1 + B_2$. Then we have $G = A+B = (A+B_2)+B_1$ and we obtain $G_1 = G_1 \cap (A+B_2)+B_1$. The direct summand $G_1 \cap (A+B_2)$ evidently contains all the elements $g_1, \dots, g_n$ and is isomorphic with complementary direct summand to B_1 in G_1, which is a completely decomposable group of finite rank.

THEOREM 3.2. *Let $\sum^*_{\mathfrak{A}\in\Omega(G)} G^{(\mathfrak{A})}$ be the canonical decomposition of the group $G = \sum^*_{t\in T} R_t$. If all the groups $G^{(\mathfrak{A})} = \sum^*_{t\in T_{\mathfrak{A}}} R_t$ $(\mathfrak{A} \in \Omega(G))$ and the group $V = \sum^*_{\mathfrak{A}\in\Omega(G)} R^{(\mathfrak{A})}$ (where $R^{(\mathfrak{A})}$ is the group of rank one and the type $\mathfrak{A}$) are separable, G is separable, too.*

Proof. Let $g_i = (g_i^{(\mathfrak{A})})_{\mathfrak{A}\in\Omega(G)}$ $(i = 1, \dots, n)$ be an arbitrary finite set of elements of the group G. Since for every $\mathfrak{A} \in \Omega(G)$ the group $G^{(\mathfrak{A})}$ is assumed to be separable, it follows by Theorem 2.1 that there exists a completely decomposable direct summand $A^{(\mathfrak{A})}$ of $G^{(\mathfrak{A})}$ of rank $n_{\mathfrak{A}} \leqslant n$ containing all the $g_i^{(\mathfrak{A})}$ $(i = 1, \dots, n)$. It is easy to show, that the complete direct sum $\sum^*_{\mathfrak{A}\in\Omega(G)} A^{(\mathfrak{A})}$ is a direct summand of G and that it contains all the elements g_i $(i = 1, \dots, n)$. Thus it remains to prove that the group $\sum^*_{\mathfrak{A}\in\Omega(G)} A^{(\mathfrak{A})}$ is separable.

Let $A^{(\mathfrak{A})} = \sum_{j=1}^{n_{\mathfrak{A}}} R_j^{(\mathfrak{A})}$ be a decomposition into a direct sum of groups of rank one. Then we have $\sum^*_{\mathfrak{A}\in\Omega(G)} A^{(\mathfrak{A})} = \sum^*_{\mathfrak{A}\in\Omega(G)} (\sum_{j=1}^{n_{\mathfrak{A}}} R_j^{(\mathfrak{A})}) = \sum_{j=1}^{n} (\sum^*_{\mathfrak{A}\in\Omega_j} R_j^{(\mathfrak{A})})$, Ω_j being the set $\Omega_j =$

*) The power of the set T is of measure zero, if there exists no σ-measure μ on the field of all subsets of the set T such that μ takes values 0 and 1 only, being 0 on the finite subsets of T and for the whole sets 1.

$= \{\mathfrak{A} \in \Omega(G); j \leqslant n_{\mathfrak{A}}\}$. Since the group $\sum\limits_{\mathfrak{A}\in\Omega(G)}^{*} R^{(\mathfrak{A})}$ is separable, then, by Theorem 3.1, the groups $\sum\limits_{\mathfrak{A}\in\Omega j}^{*} R_j^{(\mathfrak{A})}$ are separable and the group $\sum\limits_{\mathfrak{A}\in\Omega(G)}^{*} A^{(\mathfrak{A})}$ as the discrete direct sum of separable groups is also separable.

THEOREM 3.3. *If the group* $G = \sum\limits_{\mathfrak{A}\in\Omega(G)}^{*} G^{(\mathfrak{A})}$ *is separable, the group* V (*where* V *has the same meaning as in Theorem* 3.2) *and the groups* $G^{(\mathfrak{A})}$ $(\mathfrak{A} \in \Omega(G))$ *are separable.*

Proof. The separability of V follows from Theorem 3.1. If $\mathfrak{A} \in \Omega_{(0,\infty)}(G)$, then, by Theorem 2.2, $G^{(\mathfrak{A})}$ is homogeneous and, hence, separable. Assume now that $\mathfrak{A} \in \Omega_{*}(G)$ and let $G^{(\mathfrak{A})} = \sum\limits_{t\in T_{\mathfrak{A}}}^{*} R_t$. It is easy to see that the set $T_{\mathfrak{A}}$ must be finite, and therefore the group $G^{(\mathfrak{A})}$ is also separable. In fact: if $T_{\mathfrak{A}}$ is infinite, then $G^{(\mathfrak{A})}$ can be decomposed in the form $G^{(\mathfrak{A})} = G_1^{(\mathfrak{A})} + G_2^{(\mathfrak{A})}$ where $G_1^{(\mathfrak{A})} = \sum\limits_{t\in T'_{\mathfrak{A}}}^{*} R_t$ and $T'_{\mathfrak{A}}$ is of power $\aleph_0$. Since $G^{(\mathfrak{A})}$ is a direct summand of $G^{(\mathfrak{A})}$, $G_1^{(\mathfrak{A})}$ is a direct summand of G also. From Theorem 3.1 it would follow that $G_1^{(\mathfrak{A})}$ is separable which is impossible in view of Theorem 2.2.

COROLLARY 3.4. *The separability of the group G, which is of the form* $G = \sum\limits_{t\in T}^{*} R_t$ *depends only on types of axes and powers of sets of isomorphic axes.*

4. Necessary and sufficient conditions

THEOREM 4.1. *If the group* $V = \sum\limits_{\mathfrak{A}\in\Omega(G)}^{*} R^{(\mathfrak{A})}$ *is separable, then the set of types* $\Omega(G)$ *satisfies the following condition*

(W_0) *for every subset* Ω' *of the set* $\Omega(G)$ *and for every choice of characteristics* $(\chi_{\mathfrak{A}})_{\mathfrak{A}\in\Omega'}$ *there exists a finite subset* $\Omega'_0 \subset \Omega'$ *such that* $\bigcap\limits_{\mathfrak{A}\in\Omega'} \chi_{\mathfrak{A}} = \bigcap\limits_{\mathfrak{A}\in\Omega'} \chi_{\mathfrak{A}}$.

Proof. Let Ω' be an arbitrary subset of $\Omega(G)$ and $(\chi_{\mathfrak{A}})_{\mathfrak{A}\in\Omega'}$ an arbitrary choice of characteristics belonging to types $\mathfrak{A} \in \Omega'$. Since V is separable and the power of Ω' is of measure zero, we have by Theorem 3.1 that the group $V_1 = \sum\limits_{\mathfrak{A}\in\Omega'}^{*} R^{(\mathfrak{A})}$ is separable, too. Let $g = (g_{\mathfrak{A}})_{\mathfrak{A}\in\Omega'}$ be an element of V_1 such that $g_{\mathfrak{A}} \neq 0$ and $\chi(g_{\mathfrak{A}}) = \chi_{\mathfrak{A}}$ for $\mathfrak{A} \in \Omega'$. Since $g \in V_1$, there exists a completely decomposable direct summand H of finite rank of V_1 such that $g \in H$. Let $H = H_1 + \ldots + H_m$, where $H_1, \ldots, H_m$ are of rank one and of types $\mathfrak{A}^{(1)}, \ldots, \mathfrak{A}^{(m)}$, respectively. Then $g = h_1 + \ldots + h_m$ $h_j \in H_j$ $(j = 1, \ldots, m)$ and by results of A. P. Mishina [4] for each $j = 1, \ldots, m$ there exists $\mathfrak{A}_j \in \Omega'$ such that $\mathfrak{A}^{(j)} = \mathfrak{A}_j$ and, consequently, $H_j \cong R^{(\mathfrak{A}_j)}$. Hence, it follows that for every $h_j \in H_j$ there exists a $g'_{\mathfrak{A}_j} \in R^{(\mathfrak{A}_j)}$ such that $\chi(h_j) = \chi(g'_{\mathfrak{A}_j}) = \chi'_{\mathfrak{A}_j}$. Since $\chi(g) = \bigcap\limits_{\mathfrak{A}\in\Omega'} \chi(g_{\mathfrak{A}})$, then

$$\bigcap_{\mathfrak{A}\in\Omega'} \chi_{\mathfrak{A}} = \chi(g) = \bigcap_{j=1}^{m} \chi(h_j) = \bigcap_{j=1}^{m} \chi'_{\mathfrak{A}_j}, \text{ where } (\mathfrak{A}'_1, \ldots, \mathfrak{A}'_m) \subset \Omega'. \tag{1}$$

We shall prove now, that there exists a finite subset Ω'_0 of Ω' such that $(\mathfrak{A}_1, \ldots, \mathfrak{A}_m) \subset \Omega'_0$ and $\bigcap\limits_{j=1}^{m} \chi'_{\mathfrak{A}_j} = \bigcap\limits_{\mathfrak{A}\in\Omega'_0} \chi_{\mathfrak{A}}$. It is easy to see that $\bigcap\limits_{j=1}^{m} \chi'_{\mathfrak{A}_j} \leqslant \bigcap\limits_{j=1}^{m} \chi_{\mathfrak{A}_j}$ and

that the type $\tau(g)$ is determined by characteristic $\chi' = \bigcap_{j=1}^{m} \chi'_{\mathfrak{A}_j}$ and also by characteristic $\chi = \bigcap_{j=1}^{m} \chi_{\mathfrak{A}_j}$. Thus there exists at most a finite number of positive integers $\alpha_1, \ldots, \alpha_k$ such that if $\chi = (l_1, l_2, \ldots)$ and $\chi' = (l'_1, l'_2, \ldots)$, then $l'_{\alpha_i} < l_{\alpha_i} < \infty$ $(i = 1, \ldots, k)$. By (1) it follows, that for every type $\mathfrak{A} \in \Omega'$, $\mathfrak{A} \neq \mathfrak{A}_j$ $(j = 1, \ldots, m)$ we have

$$\chi_{\mathfrak{A}} \geqq \chi'. \tag{2}$$

By $\bigcap_{\mathfrak{A} \in \Omega'} \chi_{\mathfrak{A}} = \chi'$ we have that for every $i = 1, \ldots, k$ there exists a type $\mathfrak{A}'_i \in \Omega'$ such that if $\chi_{\mathfrak{A}'_i} = (l_1^{(i)}, l_2^{(i)}, \ldots)$, then $l_{\alpha_i}^{(i)} = l'_{\alpha_i}$. Hence,

$$\bigcap_{j=1}^{m} \chi'_{\mathfrak{A}_j} = \Big(\bigcap_{j=1}^{m} \chi_{\mathfrak{A}_j}\Big) \cap \Big(\bigcap_{i=1}^{k} \chi_{\mathfrak{A}'_i}\Big).$$

If we denote the subset $(\mathfrak{A}_1, \ldots, \mathfrak{A}_m, \mathfrak{A}'_1, \ldots, \mathfrak{A}'_k)$ by Ω'_0, then from (1) we have that our condition is necessary.

THEOREM 4.2. *Let the group $V = \sum^*_{\mathfrak{A} \in \Omega(G)} R^{(\mathfrak{A})}$ be separable. Then the set $\Omega(G)$ satisfies the following conditions*:

(W$_1$) *minimal condition, i.e., every descending sequence of types $(\mathfrak{A}_n)_{n=1,2,\ldots}$ of the set $\Omega(G)$ is finite*;

(W$_2$) *every subset $(\mathfrak{A}_t)_{t \in T_0}$ of incomparable types belonging to $\Omega(G)$ is finite.*

Proof. The first condition (W$_1$) follows from Theorem 4.1. In fact, let $\mathfrak{A}_1 > \mathfrak{A}_2 > \ldots$ be a descending sequence of types belonging to $\Omega(G)$. Then for every choice of characteristics $\chi_n \in \mathfrak{A}_n$ we have by Theorem 4.1 $\bigcap_{n=1}^{\infty} \chi_n = \bigcap_{j=1}^{n} \chi_{n_j} = \chi'_{n_k} \in \mathfrak{A}_{n_k}$ which is possible only if the sequence $(\mathfrak{A}_n)_{n=1,2,\ldots}$ is finite. Let now $(\mathfrak{A}_t)_{t \in T_0}$ be a subset of incomparable types of $\Omega(G)$ and let us denote $V_1 = \sum^*_{t \in T_0} R^{(\mathfrak{A}_t)}$. Then V_1 is direct summand of V and by Theorem 3.1 it is separable. Since every component $R^{(\mathfrak{A}_t)}$ $(t \in T_0)$ is fully invariant in V_1, it follows immediately that every completely decomposable direct summand H of finite rank of group V_1 coincides with direct sum of some groups $R^{(\mathfrak{A}_t)}$. Let us assume that T_0 is infinite and $g = (g_t)_{t \in T_0}$, $0 \neq g_t \in R^{(\mathfrak{A}_t)}$. Then there exists no completely decomposable direct summand H of finite rank of group V_1 containing g which contradits our assumption.

THEOREM 4.3. *The conditions* (W$_1$) *and* (W$_2$) *of Theorem* 4.2 *are sufficient for the group $V = \sum^*_{\mathfrak{A} \in \Omega'} R^{(\mathfrak{A})}$, where $\Omega' \subset \Omega_{(0,\infty)}(G)$, to be separable.*

Proof. We may suppose by (1. 2) and [3] that all $R^{(\mathfrak{A})}$ $(\mathfrak{A} \in \Omega')$ are reduced, and therefore slender. We shall prove that

(*) for every direct summand A_1 of V of finite rank, $V = A_1 + B_1$, there exists a decomposition $V = A + B$, where $A_1 \subset A$ and A is of finite rank and $B = \sum^*_{\mathfrak{A} \in \Omega' \setminus \Omega'_0} R^{(\mathfrak{A})}$, where Ω'_0 is some finite subset of Ω'.

Obviously A_1 is slender. Let h be the projection of V onto its direct summand A_1. It follows by a theorem of J. Łoś ([1], th. 47, 2), that $h(R^{(\mathfrak{A})}) = \{0\}$ for almost all $\mathfrak{A} \in \Omega'$. Let Ω'_0 be the set of all $\mathfrak{A} \in \Omega'$ for which $h(R^{(\mathfrak{A})}) \neq \{0\}$ and $A = \sum^*_{\mathfrak{A}\in\Omega'_0} R^{(\mathfrak{A})}$. Then $V = A + \sum^*_{\mathfrak{A}\in\Omega'\setminus\Omega_0} R^{(\mathfrak{A})}$ and $h(\sum^*_{\mathfrak{A}\in\Omega'\setminus\Omega'_0} R^{(\mathfrak{A})}) = \{0\}$, $\sum^*_{\mathfrak{A}\in\Omega'\setminus\Omega'_0} R^{(\mathfrak{A})} \subset B_1$. Hence, our assertion in (*) that $A_1 \subset A$ is justified.

Evidently, (*) implies, that it is sufficient to prove that every single element $g \in V$ is contained in some completely decomposable direct summand of V of finite rank.

Let g be an element of group V. We may assume that if $g = (g_{\mathfrak{A}})_{\mathfrak{A}\in\Omega'}$ then $g_{\mathfrak{A}} \neq 0$ for every $\mathfrak{A} \in \Omega'$. It follows (since (W_1) is satisfied) that in Ω' there are minimal types ($\mathfrak{A}$ is minimal in Ω', if $\mathfrak{A}' \in \Omega'$ and $\mathfrak{A}' \leqslant \mathfrak{A}$ imply $\mathfrak{A}' = \mathfrak{A}$). The set of all minimal types is finite, since any two different minimal types are incomparable and (W_2) is assumed.

Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be a set of all minimal types in Ω'. If $n = 1$, then for every $\mathfrak{A} \in \Omega'$ the inequality $\mathfrak{A} \neq \mathfrak{A}_1$ implies $\mathfrak{A}_1 < \mathfrak{A}$. Let us denote by $\{g\}_p$ the least pure subgroup of V containing g. It is easy to see that $V = \{g\}_p + U$, where $U = \sum^*_{\mathfrak{A}_1 \neq \mathfrak{A}\in\Omega'} R^{(\mathfrak{A})}$. This is a proof of our Theorem 4.3 for $n = 1$.

We shall prove this Theorem by induction on the number of minimal types. Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}_n$ be the set of minimal types belonging to Ω'. For every $\mathfrak{A} \in \Omega'$ there exists i $(i = 1, \ldots, n)$ such that $\mathfrak{A}_i \leqslant \mathfrak{A}$. Hence, $\Omega' = \Omega'_1 \cup \ldots \cup \Omega'_{n-1} \cup \Omega'_n$, where Ω'_i is the set of all $\mathfrak{A} \in \Omega'$ such that $\mathfrak{A}_i \leqslant \mathfrak{A}$.

Let $V_1 = \sum^*_{\mathfrak{A}\in\Omega'\setminus\Omega'_n} R^{(\mathfrak{A})}$, $V_2 = \sum^*_{\mathfrak{A}\in\Omega'_n} R^{(\mathfrak{A})}$. Then we have $V = V_1 + V_2$. In the set $\Omega' \setminus \Omega'_n = \Omega'_1 \cup \ldots \cup \Omega'_{n-1}$ the only minimal types are $\mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}$ and it follows by induction that every element of V_1 is contained in completely decomposable direct summand of finite rank. The same also holds for the group V_2, since the only minimal type of Ω'_n is $\mathfrak{A}_n$. The element g may be written in the form $g = g_1 + g_2$, where $g_1 \in V_1$, $g_2 \in V_2$, and therefore $V = A_1 + U_1$, $V_2 = A_2 + U_2$, where $g_1 \in A_1$, $g_2 \in A_2$, A_1 and A_2 are of finite ranks and completely decomposable.

THEOREM 4.4. *The group* $V = \sum^*_{\mathfrak{A}\in\Omega'} R^{(\mathfrak{A})}$ *where* $\Omega' \subset \Omega_*(G)$ *is separable if and only if the set* Ω' *is finite.*

Proof. Sufficiency of this Theorem is trivial. Let V be separable group. First we shall show that every increasing sequence $(\mathfrak{A}_n)_{n=1,2,\ldots}$ of types from Ω' is finite. Let $\mathfrak{A}_1 < \mathfrak{A}_2 < \ldots$ be an increasing sequence of types from Ω'. Separability of V implies separability of $V_1 = \sum^{*\infty}_{n=1} R^{(\mathfrak{A}_n)}$ by Theorem 3.1. Hence, for every $g \in V_1$ there exists a completely decomposable direct summand H of V_1 such that $H = H_1 + \ldots + H_m$, $g \in H$ and H_j $(j = 1, \ldots, m)$ being of rank one. For every $j = 1, \ldots, m$ there exists n_j such that $\tau(H_j) = \mathfrak{A}_{n_j}$ and if $g \in H$ then $\tau(g) = \mathfrak{A}_k$, where $k = \min_{1\leqslant j\leqslant m} n_j$. Thus, it follows that for every $g \in V_1$ we have $\mathfrak{A}_1 \leqslant \tau(g)$. Let us suppose that the sequence $(\mathfrak{A}_n)_{n=1,2,\ldots}$ is infinite. Then it is possible to choose an element $g = (g_n)_{n=1,2,\ldots}$, $g_n \in R^{(\mathfrak{A}_n)}$ such that $\tau(g) < \mathfrak{A}_1$. In fact: if $\mathfrak{A}_n \in \Omega'$ for $n = 1, 2, \ldots$ and $\mathfrak{A}_1 < \mathfrak{A}_2 < \ldots$, then there exists an infinite sequence of positive

integers $a_1, a_2, \ldots$ and a sequence of characteristics $\chi_n = (l_1^{(n)}, l_2^{(n)}, \ldots)$ belonging to the types $\mathfrak{A}_n$ $(n = 1, 2, \ldots)$, respectively, such that $l_{a_j}^{(n)}$ is different from 0 and ∞ for all $n = 1, 2, \ldots$ and almost all $j = 1, 2, \ldots$. Let $g = (g_n)_{n=1,2,\ldots}$, $g_n \in R^{(\mathfrak{A}_n)}$ be such an element of V_1 that the characteristic of g_n belonging to $\mathfrak{A}_n$ have zero on the a_n-th place. Then $\chi(g) = (l_1, l_2, \ldots)$ if $l_{a_j} = 0$ for every $j = 1, 2, \ldots$ and $\tau(g) < \mathfrak{A}_1$. But this least inequality contradicts our assumption. Thus we have proved that every increasing sequence of types $(\mathfrak{A}_n)_{n=1,2,\ldots} \subset \Omega_*(G)$ is finite. The separability of V implies that the conditions (W_1) and (W_2) from Theorem 4.2 are fulfilled, which proves the necessity of the condition of Theorem 4.2.

5. The main results

THEOREM 5.1. *Let* $G = \sum^*_{\mathfrak{A}\in\Omega(G)} G^{(\mathfrak{A})} = \sum^*_{\mathfrak{A}\in\Omega(G)} \sum^*_{t\in T_{\mathfrak{A}}} R_t$ *be canonical decomposition of group* $G = \sum^*_{t\in T} R_t$ *($T_{\mathfrak{A}}$ is the set of all $t \in T$ for which $\tau(R_t) = \mathfrak{A}$). The group G is separable if and only if*

(a) *the set $\Omega_{(0,\infty)}(G)$ satisfies conditoins (W_1) and (W_2) of Theorem 4.2;*

(b) *the set $\Omega_*(G) = \Omega(G) \setminus \Omega_{(0,\infty)}(G)$ is finite*

(c) *if $\mathfrak{A} \in \Omega^*(G)$, then the set $T_{\mathfrak{A}}$ is finite.*

THEOREM 5.2. *The group* $G = \sum^*_{t\in T} R_t = \sum^*_{\mathfrak{A}\in\Omega(G)} \sum^*_{t\in T_{\mathfrak{A}}} R_t$ *is separable if and only if the following conditions hold:*

(a′) *for every subset $\Omega' \subset \Omega(G)$ and every choice of characteristics $\chi_{\mathfrak{A}} \in \mathfrak{A}$ $(\mathfrak{A} \in \Omega')$ there exists a finite subset $\Omega'_0 \subset \Omega'$ such that* $\bigcap_{\mathfrak{A}\in\Omega'} \chi_{\mathfrak{A}} = \bigcap_{\mathfrak{A}\in\Omega'_0} \chi_{\mathfrak{A}}$;

(b′) *every subset $(\mathfrak{A}_t)_{t\in T_0}$ of incomparable types belonging to $\Omega(G)$ is finite;*

(c′) *for every $\mathfrak{A} \in \Omega_*(G)$ the set $T_{\mathfrak{A}}$ is finite.*

Proof. Conditions (W_1) and (W_2) of Theorem 4.2 follow from (a′) and (b′). These conditions are sufficient for the separability of groups $\sum^*_{\mathfrak{A}\in\Omega'} R^{(\mathfrak{A})}$, where $\Omega' \subset \Omega_{(0,\infty)}(G)$. To prove this it is sufficient to show that if (a′) and (b′) hold, then the set $\Omega_*(G)$ is finite, i.e., it is sufficient to prove that every increasing sequence $(\mathfrak{A}_n)_{n=1,2,\ldots}$ of the types belonging to $\Omega_*(G)$ is finite.

Let $\mathfrak{A}_1 < \mathfrak{A}_2 < \ldots$ be an infinite increasing sequence of types from $\Omega_*(G)$. In this case it is possible to choose an infinite sequence of characteristics $(\chi_n)_{n=1,2,\ldots}$ such that $\chi_n \in \mathfrak{A}_n$ $(n = 1, 2, \ldots)$ and $\bigcap_{n=1} \chi_n = \chi'$, where $\chi' \in \mathfrak{A}' < \mathfrak{A}_1$, which contradicts (a′), (see proof of Theorem 4.4). Hence, every sequence $\mathfrak{A}_1 < \mathfrak{A}_2 < \ldots$, where $\mathfrak{A}_n \in \Omega_*(G)$ $(n = 1, 2, \ldots)$ is finite and, consequently, the set $\Omega_*(G)$ must be finite.

THEOREM 5.3 *If the group* $G = \sum^*_{t\in T} R_t$ *is separable, then for every subset $T' \subset T$ the group* $G' = \sum^*_{t\in T'} R_t$ *is separable, too.*

Proof. It follows from Theorem 5.1 or Theorem 5.2.

NICHOLAS COPERNICUS UNIVERSITY, TORUŃ
(UNIWERSYTET MIKOŁAJA KOPERNIKA, TORUŃ)

REFERENCES

[1] L. Fuchs, *Abelian groups*, Budapest, (1958).

[2] R. Baer, *Abelian groups without elements of finite order*, Duke Math. Journ. **3** (1937), 68—122.

[3] E. Sąsiada, *Proof that every countable and reduced torsion-free Abelian group is slender*, Bull. Acad. Polon. Sci. Sér. sci. math., astr. et phys. **7** (1959), 143—144.

[4] A. P. Mishina, *On the isomorphism of complete direct sums of Abelian groups of rank one without torsion* [in Russian], Mat. Sb., **31** (1952), 118—127.

[5] E. Sąsiada, *On the isomorphism of decomposition of torsion-free Abelian groups into complete direct sums of groups of rank one*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys. **7** (1959), 145—149.

[6] M. Król and E. Sąsiada, *The complete direct sums of torsion-free Abelian groups of rank one which are separable*, ibid. **8** (1960), 1—2.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 5, 1961

MATHEMATICS

An AR-Set with an Infinite Number of $\mathcal{R}$-Neighbours

by

K. BORSUK

Presented on March 11, 1961

The present note is a contribution to the problem of the classification of topological spaces from the point of view of the theory of retracts. As in [1] and [2], we consider the relations $\underset{\mathcal{R}}{\leqslant}$ and $\underset{\mathcal{R}}{<}$, defined for arbitrary spaces X, Y as follows:

$X \underset{\mathcal{R}}{\leqslant} Y$ means that H is homeomorphic to a retract of Y,

$X \underset{\mathcal{R}}{=} Y$ means that both relations $X \underset{\mathcal{R}}{\leqslant} Y$ and $Y \underset{\mathcal{R}}{\leqslant} X$ hold,

$X \underset{\mathcal{R}}{<} Y$ means that $X \underset{\mathcal{R}}{\leqslant} Y$, but $Y \underset{\mathcal{R}}{\leqslant} X$ does not hold.

If the relation $X \underset{\mathcal{R}}{\ } Y$ does not hold, then we say that X and Y are *$\mathcal{R}$-distinct*. If none of the relations $X \underset{\mathcal{R}}{\leqslant} Y$ or $Y \underset{\mathcal{R}}{\leqslant} X$ holds, then X and Y are said to be *$\mathcal{R}$-incomparable*.

If $X \underset{\mathcal{R}}{<} Y$, but not for any space Z it is $X \underset{\mathcal{R}}{<} Z \underset{\mathcal{R}}{<} Y$, then X is said to be an *$\mathcal{R}$-neighbour of Y on the left*, and Y — an *$\mathcal{R}$-neighbour of X on the right*.

Evidently, if X is an ANR-set (respectively an AR-set) then every space $Y \underset{\mathcal{R}}{\leqslant} X$ is also an ANR-set (respectively an AR-set), but not conversely. Even if X is a polytope (- space homeomorphic with a compact geometric polyhedron) then under its $\mathcal{R}$-neighbours on the right may appear spaces with a rather complicated structure (see [2] pp. 459 and 460). However, the following problem remains open:

PROBLEM 1. *Does there exist a polytope P with an infinite number of $\mathcal{R}$-distinct $\mathcal{R}$-neighbours on the left?*

PROBLEM 2. *Does there exist a polytope P for which there exists an infinite number of $\mathcal{R}$-distinct ANR-sets being $\mathcal{R}$-neighbours on the left of P?*

There exist polytopes having under their $\mathcal{R}$-neighbours on the left sets $\mathcal{R}$-distinct from polytopes ([2], p. 460). However, the following problem remains open

PROBLEM 3. *Is every ANR-set being an $\mathcal{R}$-neighbour on the right of a polytope $\mathcal{R}$-equal to a polytope?*

The purpose of the present note is to prove the following

THEOREM. *There exists in the 3-dimensional Euclidean space E^3 a 2-dimensional AR-set X such that*:

1° *X has an infinite set of $\mathcal{R}$-distinct $\mathcal{R}$-neighbours on the left.*

2° *There exist $2^{\aleph_0}$ $\mathcal{R}$-distinct AR-sets which are $\mathcal{R}$-neighbours on the right of X.*

Proof. We start with a construction given in [1], p. 325. Let A_n denote a polytope in E^3 consisting of $n+2$ disks $D_0, D_1, \dots, D_{n+1}$ satisfying the following conditions: The centres of disks $D_1, D_2, \dots, D_n$ lay on a straight line $L \subset E^3$ and two subsequent disks have only one point in common, while the not subsequent — are disjoint. Let a and b denote the endpoints of the segment $L \cap \bigcup_{i=1}^{n} D_i$. The disks D_0 and D_{n+1} lay in planes perpendicular to L and a is the centre of D_0, and b is the centre of D_{n+1}. The disks D_0 and D_{n+1} are said to be the *lower* and the *upper base* of the polytope A_n respectively. It is easy to observe (see [1], p. 326) that $m < n$ implies $A_n \underset{R}{<} A_m$.

Now let B_n denote the polytope (in E^3) obtained from n disjoint copies of the polytope A_n by the identification of their lower bases.

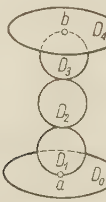

Fig. 1.

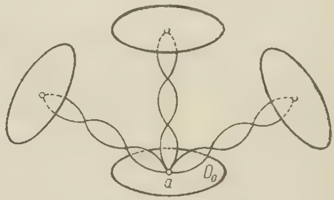

Fig. 2.

Thus, we get a sequence $\{B_n\}$ of polytopes such that no one of them is homeomorphic to a subset of the other (see [1], p. 326). Manifestly, everyone of the polytopes B_n is 2-dimensional at every of its points.

Now let us consider in E^3 the segment J with endpoints $(0, 0, 0)$ and $(1, 0, 0)$ and let $\{r_n\}$ be the sequence of all (distinct) rational numbers belonging to the open interval $(0, 1)$. For every $n = 1, 2, \dots$ let J_n denote the segment with the endpoints $(r_n, 1/n, 0)$ and $(r_n, -1/n, 0)$ and Q_n denote a ball (in E^3) with centre $(r_n, 1/n, 0)$ and radius ϱ_n, where $0 < \varrho_n < 1/n$ is chosen in such a manner that the sets $J_n \cup Q_n$ are disjoint one to the other. It follows, in particular, that $\varrho_n \to 0$. Evidently, there exists a homemorphism h_n mapping B_n into a subset of Q_n containing the point $(r_n, 1/n - \varrho_n, 0)$. One sees easily that the set

$$X = \bigcup_{n=1}^{\infty} [(J_n - Q_n) \cup h_n(B_n)] \cup J$$

is an AR-set.

Consider the subset Y of X made up of points $y \in X$ at which X has dimension 2. Evidently,

(1) The components of Y coincide with the sets $h_n(B_n)$.

(2) For every point p of J there exists an increasing sequence of indices $\{n_k\}$ such that $h_{n_k}(B_{n_k}) \to (p)$.

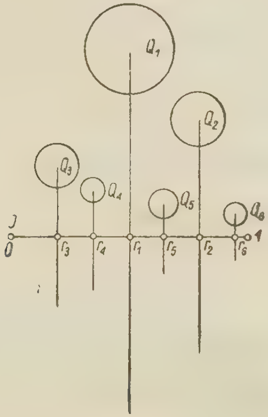

Fig. 3.

Now let us prove the following

LEMMA. *If* $M_1 = Y \cup N_1$, $M_2 = Y \cup N_2$ *are two compacta such that* $\dim N_1 \leqslant 1$ *and* $\dim N_2 \leqslant 1$ *and if* h *is a homeomorphism mapping* M_1 *into* M_2, *then*

(3) $hh_n(B_n) \subset h_n(B_n)$ *for every* $n = 1, 2, \ldots$,

(4) $h(p) = p$ *for every point* $p \in J$.

Proof of the Lemma. It follows by our hypotheses that Y coincides with the set of points at which M_1 has the dimension 2, and also with the set of points at which M_2 has dimension 2. We infer that h maps each component of Y into a component of Y. From (1) it follows that for every $n = 1, 2, \ldots$ there exists a natural m such that $h[h_n(B_n)] \subset h_m(B_m)$. If we recall that none of the sets B_n is homeomorphic with a subset of another set $B_{n'}$, we conclude that $n = m$, i.e.

$$h[h_n(B_n)] \subset h_n(B_n) \text{ for every } n = 1, 2, \ldots$$

It remains to apply (2) to complete the proof.

Proof of the Theorem. Ad 1°. For a given natural n let us denote by X_n the set which we get, if we remove from X the half-open segment consisting of all points $(r_n, t, 0)$ with $-1/n \leqslant t \leqslant 0$. Evidently, X_n is a retract of X, consequently

(5) $$X_n \underset{\mathcal{R}}{\leqslant} X \text{ for every } n = 1, 2, \ldots$$

On the other hand, X is not homeomorphic to any subset of X_n. In fact, would h be a homeomorphism of X into X_n then we would infer by the lemma (setting $M_1 = X$ and $M_2 = X_n$) that h maps the point $p_n = (r_n, 0, 0)$ onto itself. But this is impossible, because the order of this point in X is equal to 4, and in X_n only to 3.

It follows by (3) that

(6) $$X_n \underset{\mathcal{R}}{<} X \text{ for every } n = 1, 2, \dots$$

By a quite analogous argument we infer that

(7) For $n \neq m$ the sets X_n and X_m are $\mathcal{R}$-distinct.

Now let us consider a space Z such that

(8) $$X_n \underset{\mathcal{R}}{\leqslant} Z \underset{\mathcal{R}}{\leqslant} X.$$

We may assume that $Z \subset X$. The inequality $Z \underset{\mathcal{R}}{\leqslant} X$ implies that Z is an AR-set. If $Z \underset{\mathcal{R}}{\subset} X_n$, then Z is a retract of X_n and we infer that $Z \underset{\mathcal{R}}{=} X_n$. If, however, $Z - X_n \neq 0$, then

$$Z = (X_n \cap Z) \cup J',$$

where $X_n \cap Z$ is an AR-set and J' is a segment made up of points $(r_n, t, 0)$ with $s_n \leqslant t \leqslant 0$, where s_n is a number satisfying the inequality $-1/n \leqslant s_n < 0$. Since $X_n \underset{\mathcal{R}}{\leqslant} Z$, there exists a homeomorphism g mapping X_n into Z. Applying the Lemma, we infer by (3) and (4) that g maps each component of the set $X_n - J$ into itself and that

$$g(p) = p \text{ for every point } p \in J.$$

It follows that g maps X_n onto $Z \cap X_n$. If we set

$$g'(p) = g(p) \text{ for every point } p \in X_n,$$

$$g'(r_n, t, 0) = (r_n, -ns_n t, 0) \quad \text{for } -1/n \leqslant t \leqslant 0,$$

we get a homeomorphism g' mapping X onto Z. If follows that $X \underset{\mathcal{R}}{\leqslant} Z$, and, consequently, $X \underset{\mathcal{R}}{=} Z$.

Thus we have shown that (8) implies that $X_n \underset{\mathcal{R}}{=} Z$ or $X \underset{\mathcal{R}}{=} Z$. With regard to (6) and (7), we infer that the sets $X_1, X_2, \dots$ are $\mathcal{R}$-distinct $\mathcal{R}$-neighbours on the left of X.

Ad 2°. Let t be an irrational number of the interval $\langle 0,1 \rangle$ and let J_t denote the segment made up of the points $(t, u, 0)$ with $-1 \leqslant u \leqslant 0$. Evidently, the set

$$X_t = X \cup J_t$$

is an AR-set such that $X \underset{\mathcal{R}}{\leqslant} X_t$. By the same arguments as used in the proof of 1°, we prove, step by step, that $X \underset{\mathcal{R}}{<} X_t$ and that for $t_1 \neq t_2$ the spaces X_{t_1} and X_{t_2} are $\mathcal{R}$-distinct. Also the proof that X_t is an $\mathcal{R}$-neighbour on the right of X is quite analogous to the proof that X is an $\mathcal{R}$-neighbour on the right of the set X_n. Since the parameter t runs through all irrational numbers of the interval $\langle 0,1 \rangle$ we conclude that the family $\{X_t\}$ consists of $2^{\aleph_0}$ $\mathcal{R}$-distinct AR-sets which are $\mathcal{R}$-neighbours of X on the right. Thus, the proof of the Theorem is concluded.

Problem 4. *Does there exist an AR-set with $2^{\aleph_0}$ $\mathcal{R}$-distinct $\mathcal{R}$-neighbours on the left?*

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY, PAN)

REFERENCES

[1] K. Borsuk, *Concerning the classification of topological spaces from the stand-point of the theory of retracts*, Fund. Math., **46** (1959), 321—330.

[2] — , *Concerning the notion of $\mathcal{R}$-neighbours*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **8** (1960), 459—462.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 5, 1961

MATHÉMATIQUE

Propriétés d'une intégrale singulière dans l'espace

par

W. POGORZELSKI

Présenté par T. WAŻEWSKI le 14 mars, 1961

Soit dans l'espace euclidien E_n à n dimensions un ensemble fini de $p+1$ surfaces fermées de Liapounoff $S_0, S_1, ..., S_p$ à $n-1$ dimensions ($p \geqslant 0$). On admet que ces surfaces n'ont pas de points communs et que les surfaces $S_1, ..., S_p$ sont situées d'une façon arbitraire à l'intérieur du domaine borné par la surface S_0*). Désignons par Ω un ensemble de tous les points du domaine Ω_0 non situés sur les surfaces $S_1, ..., S_p$. Cet ensemble est une somme $\Omega = \Omega_1 + \Omega_2 + ... + \Omega_m$ de domaines disjoints $\Omega_1, ..., \Omega_m$ simplement ou multiplement connexes. Dans le cas de l'absence des surfaces $S_1, ..., S_p$ on a $\Omega = \Omega_0$.

Dans cette communication nous présenterons quelques propriétés d'une intégrale singulière

$$\Phi(x, u) = \int_{\Omega} F(x-y, y, u)\, dy \tag{1}$$

qui est une généralisation de l'intégrale singulière étudiée dans notre travail [1].

Nous admettons les suppositions suivantes.

I. La fonction $F(x, y, u)$ est définie dans la région $[x \in E_n - \theta,\ y \in \Omega,\ u \in \Omega^*]$

$$F(x, y, u) = \frac{K(x', y, u)}{|x|^n}; \qquad (|x| \neq 0), \tag{2}$$

où x' est une projection centrale du point arbitraire $x \in E_n$ sur la surface ω d'une sphère unitaire de centre $\theta = (0, ..., 0)$, donc $x = x'\,|x|$.

II. La fonction réelle $K(x', y, u)$ est définie dans la région $[x' \in \omega,\ y \in \Omega,\ u \in \Omega^*]$, Ω^* étant un domaine donné dans l'espace E_n. Cette fonction vérifie les inégalités suivantes

$$|K(x', y, u)| < \frac{M_F}{|y-y_S|^{\alpha}} \tag{3}$$

$$|K(x', y, u) - K(\tilde{x}', \tilde{y}, \tilde{u})| < k_F\left[\frac{|x'-\tilde{x}'|^{h_\omega}}{|y-y_S|^{\alpha}} + \frac{|y-\tilde{y}|^{h}+|u-\tilde{u}|^{h}}{|y-y_S|^{\alpha+h}}\right]. \tag{3'}$$

*) Les considérations qui suivent s'appliquent aussi au cas du domaine Ω_0 multiplement connexe, limité par un ensemble de surfaces $S_0^{(1)}, ..., S_0^{(r)}$.

$(x', \tilde{x}')$ étant les deux points quelconques de la surface ω,

$(y, \tilde{y})$ — les deux points quelconques situés simultanément à l'intérieur d'un domaine d'entre les domaines $\Omega_1, \ldots, \Omega_m$,

$(u, \tilde{u})$ — les deux points quelconques du domaine Ω^*; nous désignons par $|x' - \tilde{x}'|$, $|y - \tilde{y}|$, $|u - \tilde{u}|$ les distances euclidiennes de ces points. On admet que les exposants constants α, h, h_ω, h_1 vérifient les inégalités

$$(4) \qquad \begin{cases} 0 < \alpha < 1; \quad 0 < h < 1; \quad \alpha + h < 1; \quad 0 < h_\omega \leqslant 1; \\ 0 < h_1 \leqslant 1; \quad h < \min(h_\omega, h_1); \end{cases}$$

M_F et k_F sont des constantes positives. Nous désignons par $|y - y_S|$ la distance entre le point $y \in \Omega$ et le point y_S le plus rapproché sur les surfaces $S_1, S_2, \ldots, S_p$; on suppose que $|y - y_S| \leqslant |\tilde{y} - \tilde{y}_S|$. La fonction $K(x', y, u)$ appartient donc à la classe $\mathfrak{H}_\alpha^h$ relativement à la variable y, [1].

III. La fonction donnée $K(x', y, u)$ vérifie l'égalité

$$(5) \qquad \int_\omega K(x', y, u)\, dx' = 0$$

quels que soient les points y et u dans les domaines Ω et Ω^*. dx' désigne l'élément d'aire d'un arc de la surface sphérique ω au point x'.

IV. La différence $K(x', y, u) - K(x', \tilde{y}, u)$ vérifie l'inégalité

$$(5') \qquad |[K(x', y, u) - K(x', \tilde{y}, u)] - [K(\tilde{x}', y, u) - K(\tilde{x}', \tilde{y}, u)]| < \frac{k_F |x' - \tilde{x}'|^{h_\omega} |y - \tilde{y}|^h}{|y - y_S|^{\alpha + h}}$$

x' et $\tilde{x}'$ étant les deux points arbitraires de la surface ω, y et $\tilde{y}$ les deux points arbitraires, situés simultanément à l'intérieur du même domaine $\Omega_1, \Omega_2, \ldots, \Omega_m$; on suppose en outre que $|y - y_S| \leqslant |\tilde{y} - \tilde{y}_S|$.

Remarquons que la propriété (5') est celle dont jouit p. ex. la fonction de la forme

$$K(x', y, u) = \sum_\nu N_\nu(x', u) f_\nu(y, u)$$

où $N_\nu(x', u)$ sont des fonctions vérifiant la condition de Hölder sur la surface ω et $f_\nu(y, u)$ sont des fonctions de classe $\mathfrak{H}_\alpha^h$ par rapport à la variable y (voir [1]),

De la propriété (5') jouit aussi la fonction $K(x', y, u)$ admettant les dérivées par rapport aux coordonnées du point $x' \in \omega$ appartenant à la classe $\mathfrak{H}_\alpha^h$ relativement à la variable y.

Nous allons démontrer les propriétés suivantes de l'intégrale singulière (1).

Théorème. *Si la fonction donnée $F(x, y, u)$ vérifie les conditions I, II, III, IV, alors l'intégrale singulière au sens de Cauchy*

$$(6) \qquad \Phi(x, u) = \int_\Omega F(x - y, y, u)\, dy = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon(|x - y| > \varepsilon)} F(x - y, y, u)\, dy$$

est une fonction d'un couple de points (x, u), *définis dans la région* $[x \in \Omega; u \in \Omega^*]$, *vérifiant dans cette région les inégalités*

$$|\Phi(x, u)| < (C_1 M_F + C_2 k_F) |x - x|^{-\alpha}, \tag{7}$$

$$|\Phi(x, u) - \Phi(\widetilde{x}, \widetilde{u})| < (C_1' M_F + C_2' k_F) \frac{|x - \widetilde{x}|^h + |u - \widetilde{u}|^h}{|x - x_S|^{\alpha+h}} \tag{7'}$$

où $C_1 C_2, C_1', C_2'$ *sont des constantes positives, indépendantes de la fonction* F, x *et* $\widetilde{x}$ *— les deux points arbitraires situés simultanément à l'intérieur d'un même domaine d'entre les domaines* $\Omega_1, \Omega_2, \dots, \Omega_m$. *En accord avec la définition donnée dans notre travail* [1], *la fonction* (6) *appartient donc à la classe* $\mathfrak{H}_\alpha^h$, *relativement à la variable* x.

La démonstration de la propriété (7) est analogue à celle donnée dans notre travail [1], donc nous l'omettons ici. Pour démontrer la propriété (7'), remarquons que l'integrale (6) est une somme d'intégrales étendues aux domaines $\Omega_1, \dots, \Omega_m$, il suffit donc d'étudier l'une d'elles

$$J(x, u) = \int_{\Omega_\nu} F(x - y, y, u)\, dy \tag{8}$$

$(\nu = 1, 2, \dots, m)$. Supposons d'abord que $\widetilde{u} = u \in \Omega^*$ et que les points $x, \widetilde{x}$ sont situés dans le domaine d'intégration Ω_ν. Nous n'étudierons que le cas $|x - \widetilde{x}| \leqslant \frac{1}{4} |x - x_S|$, puisque dans le cas contraire la propriété (7') résulte immédiatement de la propriété (7). Décomposons les valeurs de l'intégrale (8) aux points $\widetilde{x}$ et x en sommes

$$\begin{cases} J(x, u) = J^{\Pi_1}(x, u) + J^{\Pi - \Pi_1}(x, u) + J^{\Omega_\nu - \Omega}(x, u) \\ J(\widetilde{x}, u) = J^{\Pi_2}(\widetilde{x}, u) + J^{\Pi - \Pi_2}(\widetilde{x}, u) + J^{\Omega_\nu - \Pi}(x, u) \end{cases} \tag{9}$$

d'intégrales étendues aux domaines Π_1, $\Pi - \Pi_1$, $\Omega_\nu - \Pi$ etc., où Π_1 est une sphère, centrée en x, de rayon $2|x - \widetilde{x}|$ et Π_2 — une sphère, centrée en $\widetilde{x}$, de rayon $3|x - \widetilde{x}|$. Nous aurons

$$J^{\Pi_1}(x, u) = \lim_{\varepsilon \to 0} \Big\{ \int_{\Pi_1^{(\varepsilon)}} [F(x - y, y, u) - F(x - y, x, u)]\, dy + \\ + \int_{\Pi_1^{(\varepsilon)}} F(x - y - x, u)\, dy \Big\} \tag{10}$$

$\Pi_1^{(\varepsilon)}$ étant un ensemble de tous les points y de la sphère Π_1 pour lesquels $|x - y| > \varepsilon$. Il en résulte l'intégrale à la singularité faible

$$J^{\Pi_1}(x, u) = \int_{\Pi_1} [F(x - y, y, u) - F(x - y, x, u)]\, dy, \tag{11}$$

puisque, d'après la supposition III (5), l'égalité

$$\int_{\Pi_1^{(\varepsilon)}} F(x - y, x, u)\, dy = 0; \qquad 0 < \varepsilon < 2\, |x - \widetilde{x}|$$

est vraie quelque soit le centre x de la sphère $\Pi_1 \subset \Omega_\nu$ et le point u fixé dans Ω^*. En appliquant les suppositions I. (2) et II (3') à la différence dans la formule (11) et

en remarquant que la distance $|y - y_{\Pi_1}|$ du point y de la surface de la sphère Π_1 est inférieure aux distances $|x - x_S|$ et $|y - y_S|$, nous aurons

$$(12) \qquad |J^{\Pi_1}(x,u)| < \int\limits_{\Pi_1} \frac{k_F\, dy}{|x-y|^{n-h}\,|y-y_{\Pi_1}|^{a+h}} <$$

$$< k_F \int\limits_0^{2|x-\widetilde{x}|} \frac{\varrho^{n-1}\, d\varrho}{\varrho^{n-h}(|x-x_S|-\varrho)^{a+h}} < \frac{\text{const}\; k_F |x-\widetilde{x}|^h}{|x-x_S|^{a+h}}$$

et une limitation analogue pour l'intégrale $J^{\Pi_2}(x, u)$.

Nous pouvons écrire ensuite, en vertu de la supposition III (5),

$$(13) \qquad J^{\Pi-\Pi_1}(x,u) = \int\limits_{\Pi-\Pi_1} [F(x-y,y,u) - F(x-y,x,u)]\, dy =$$

$$= \Big[\int\limits_{\Pi-\Pi_2'} + \int\limits_{\Pi_2'-\Pi_2} + \int\limits_{\Pi_2-\Pi_1}\Big] [F(x-y,y,u) - F(x-y,x,u)]\, dy$$

$$(13') \qquad J^{\Pi-\Pi_2}(\widetilde{x},u) = \int\limits_{\Pi-\Pi_2} F(\widetilde{x}-y,y,u)\, dy = \int\limits_{\Pi-\Pi_2'} F(\widetilde{x}-y,y,u)\, dy +$$

$$+ \int\limits_{\Pi_2'-\Pi_2} [F(\widetilde{x}-y,y,u)\, dy - F(\widetilde{x}-y,x,u)]\, dy$$

Π_2' étant une sphére, centrée en $\widetilde{x}$, de rayon $|x-x_S| - |x-\widetilde{x}|$. Il en résulte

$$(14) \qquad J^{\Pi-\Pi_1}(x,u) - J^{\Pi-\Pi_2}(\widetilde{x},u) =$$

$$= \Big[\int\limits_{\Pi-\Pi_2'} + \int\limits_{\Pi_2-\Pi_1}\Big] \frac{K[(x-y)',y,u)] - K[(x-y),x,u]}{|x-y|^n}\, dy +$$

$$- \int\limits_{\Pi-\Pi_2'} \frac{K[(\widetilde{x}-y)',y,u]\, dy}{|\widetilde{x}-y|^n} + \int\limits_{\Pi_2'-\Pi_2} \Big\{ \frac{K[(x-y),y,u] - K[(x-y)',x,u]}{|x-y|^n} -$$

$$- \frac{K[(\widetilde{x}-y)',y,u] - K[(\widetilde{x}-y)',x,u]}{|\widetilde{x}-y|^n} \Big\}\, dy.$$

En vertu de la supposition II (3), nous aurons les limitations

$$(15) \qquad \Big|\int\limits_{\Pi-\Pi_2'} \frac{K[(x-y)',y,u]}{|x-y|^n}\, dy\Big| < \text{const}\; M_F \int\limits_{|x-a_S|-2|x-\widetilde{x}|}^{|x-x_S|} \frac{\varrho^{n-1}\, d\varrho}{\varrho^n(|x-x_S|-\varrho)^a} <$$

$$< \frac{\text{const}\; M_S}{|x-x|^a} \int\limits_{1-2\frac{|x-\widetilde{x}|}{|x-x_S|}}^{1} \frac{dt}{t(1-t)^a} < \frac{\text{const}\; M_F |x-\widetilde{x}|^h}{|x-x_S|^{a+h}}$$

$$(16) \qquad \Big|\int\limits_{\Pi-\Pi_2'} \frac{K[(x-y)',x,u]}{|x-y|^n}\, dy\Big| < \frac{\text{const}\; M_F}{|x-x_S|^a} \int\limits_{1-2\frac{|x-\widetilde{x}|}{|x-x_S|}}^{1} \frac{dt}{t} < \frac{\text{const}\; M_F |x-\widetilde{x}|^h}{|x-x_S|^{a+h}}$$

et les limitations analogues pour le point $\widetilde{x}$. Il reste à limiter la dernière des intègrales D (14). Dans ce but, en appliquant les suppositions II (3′), III (5′) et les inégalités $\frac{1}{2} < |x-y|/|\widetilde{x}-y| < 2$, vraies si $y \in \Pi - \Pi_1$, on trouve par un calcul élémentaire

$$(17)\qquad |D| < \operatorname{const} k_F \Bigg[\int\limits_{\Pi-\Pi_1} \frac{|x-\widetilde{x}|\, dy}{|x-y|^{n+1-h}\, |y-y_\Pi|^{\alpha+h}} - \int\limits_{\Pi-\Pi_1} \frac{\theta^{h_\omega}\, dy}{|x-y|^{n+h}\, |y-y_\Pi|^{\alpha+h}} \Bigg],$$

θ désignant l'angle entre les vecteurs $x-y$ et $\widetilde{x}-y$, pour lequel $\max \theta = \operatorname{arc}\sin \dfrac{|x-\widetilde{x}|}{|x-y|}$, si les distances $|x-x|$ et $|\widetilde{x}-y|$ sont constantes. Nous en concluons par un calcul analogue au précédent

$$(17')\qquad |D| < \frac{\operatorname{const} k_F\, |x-\widetilde{x}|^h}{|x-x_S|^{\alpha+h}}$$

en tenant compte de la supposition (4). Pour la différence des troisièmes intégrales régulières dans les sommes (9), on applique simplement l'inégalité (3′). Alors nous aurons

$$(18)\qquad |J^{\Omega_\nu-\Pi}(x,u) - J^{\Omega_\nu-\Pi}(\widetilde{x},u)| < \operatorname{const} k_F\, |x-\widetilde{x}|^{h_\omega} \int\limits_{\Omega_\nu-\Pi} \frac{dy}{|x-y|^{n+h_\omega}\, |y-y_S|^{\alpha}} < \frac{\operatorname{const} k_F\, |x-\widetilde{x}|^{h_\omega}}{|x-y|^{\alpha+h_\omega}}.$$

En rapprochant les inégalités (12), (15), (16), (17′), on arrive à la propriété cherchée de l'intégrale (8)

$$(19)\qquad |J(x,u) - J(\widetilde{x},u)| < (c_1' M_F + c_2' k_F)\, |x-x_S|^{-\alpha-h}\, |x-\widetilde{x}|^h$$

si $x, \widetilde{x} \in \Omega_\nu$. Dans le cas où les points x, $\widetilde{x}$ sont situés à l'extérieur du domaine Ω_ν, on doit étudier la différence des intégrales (19) par le même procédé que la différence (18).

En somme, la thèse (7) de notre Théorème se trouve démontrée, si $\widetilde{u} = u$. Il nous reste à étudier la différence $J(x,u) - J(x,\widetilde{u})$ où $x \in \Omega$; u, $\widetilde{u} \in \Omega^*$. Admettons d'abord que $x \in \Omega_\nu$ et remarquons qu'il suffit d'étudier le cas $|u-\widetilde{u}| < \frac{1}{2}|x-x_S|$. Décomposons l'intégrale (8) en somme

$$(20)\qquad J(x,u) = J^{\Lambda}(x,u) + J^{\Pi-\Lambda}(x,u) + J^{\Omega_\nu-\Pi}(x,u)$$

d'intégrales étendues: 1) à la sphère Λ, centrée en x, de rayon $|u-\widetilde{u}|$, 2) à la partie $\Pi-\Lambda$ de la sphère Π, centrée en x, de rayon $|x-x_S|$, 3) à la partie extérieure $\Omega_\nu - \Pi$.

Nous aurons, comme précédemment (voir (11)),

$$(21)\qquad |J^{\Lambda}(x,u)| < \int\limits_{\Lambda} \frac{k_F\, dy}{|x-y|^{n-h}\, |y-y_\Lambda|^{\alpha+h}} < \frac{\operatorname{const} k_F\, |u-\widetilde{u}|^h}{|x-x_S|^{\alpha+h}}$$

et une limitation analogue pour le point $\tilde{u}$. Nous écrirons ensuite, en égard à la supposition II (3′),

$$(22)\qquad |J^{\Pi-\Lambda}(x,u)-J^{\Pi-\Lambda}(x,\tilde{u})| < k_F \int\limits_{\Pi-\Lambda} \frac{|u-\tilde{u}|^{h_1}\,dy}{|x-y|^n\,|y-y_\Pi|^{\alpha+h}} <$$

$$< \frac{\mathrm{const}\,k_F\,|u-\tilde{u}|^{h_1}}{|x-x_F|^{\alpha+h}} \int\limits_{|u-\tilde{u}|/|x-x_S|}^{1} \frac{dt}{t(1-t)^{\alpha+h}} < \frac{\mathrm{const}\,k_F\,|u-\tilde{u}|^{h}}{|x-x_S|^{\alpha+h}}$$

en remarquant qu'on a supposé $h_1 < h$ et $|u-\tilde{u}| < \frac{1}{2}|x-x_S|$. D'une façon analogue on obtient une limitation de la même forme pour la différence concernant la troisième composante de la somme (20) et une limitation de la même forme (22) pour la différence $J(x,u)-J(x,\tilde{u})$, si $x \in \Omega-\Omega_\nu$. En réunissant les résultats obtenus, on arrive à la conclusion (7′) de notre Théorème.

INSTITUT MATHÉMATIQUE, ACADÉMIE POLONAISE DES SCIENCES
(INSTYTUT MATEMATYCZNY, PAN)

OUVRAGES CITÉS

[1] W. Pogorzelski, *Sur une classe de fonctions discontinues et une intégrale singulière dans l'espace*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **8** (1960), 445—451.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 5, 1961

MATHÉMATIQUE

Probléme linéaire aux dérivées tangentielles discontinues pour une fonction harmonique dans l'espace

par

W. POGORZELSKI

Présenté par T. WAŻEWSKI le 22 mars 1961

1. Énoncé du problème

Soit dans l'espace euclidien E^n à n dimensions ($n \geqslant 3$) un ensemble fini de p disques plans disjoints ($p \geqslant 1$) $D_1, D_2, \ldots, D_p$ à $n - 1$ dimensions. Tout disque D_ν ($1 \leqslant \nu \leqslant p$) est formé d'un domaine ouvert $\Omega_\nu^{(0)}$ plan à $n - 1$ dimensions limité par une frontière $S_\nu^{(0)}$ qui se compose d'une ou de quelques surfaces fermées à $n - 2$ dimensions. Soit à l'intérieur du domaine $\Omega_\nu^{(0)}$ un ensemble fini (qui peut être vide) de surfaces $S_\nu^{(1)}, S_\nu^{(2)}, \ldots, S_\nu^{(q_\nu)}$ à $n - 2$ dimensions n'ayant pas de points communs; nous désignons

$$\Omega_\nu = \Omega_\nu^{(0)} - \sum_{\gamma=1}^{q_\nu} S_\nu^{(\gamma)}. \tag{1}$$

Cet ensemble de points est une somme $\Omega_\nu = \Omega_\nu^{(1)} + \ldots + \Omega_\nu^{(q'_\nu)}$ de domaines disjoints. Dans un cas particulier on peut avoir $\Omega_\nu = \Omega_\nu^{(0)}$.

Nous posons le problème de recherche d'une fonction harmonique $u(x)$ dans le domaine illimité $E^n - \sum_{\nu=1}^{p} D_\nu$, nulle à l'infini, qui en tout point $y \in \Omega_\nu$ de tout disque D_ν ($\nu = 1, \ldots, p$) vérifie une relation linéaire

$$\frac{du}{dN_y} + \sum_{j=1}^{r_\nu} a_\nu^{(j)}(y) \frac{du(y)}{ds^{(j)}(y)} + f(y) = 0, \tag{2}$$

où 1) du/dN_y désigne une valeur limite au point y de la dérivée normale du côté *distingué* de tout disque D_ν; 2) $du/ds_\nu^{(j)}(y)$ désigne une valeur limite (independante du côté du disque) de la dérivée tangentielle dans la direction $s_\nu^{(j)}(y)$:

$$\frac{du(y)}{ds_\nu^{(j)}(y)} = \lim_{x \to y} \frac{du(x)}{ds_\nu^{(j)}(y)};$$

$s_\nu^{(1)}(y), \ldots, s_\nu^{(r_\nu)}(y)$ étant $r_\nu < n$ directions dans le plan du disque D_ν déterminées en tout point y de la région du domaine Ω_ν $(\nu = 1, 2, \ldots, p)$; 3) $a_\nu^{(1)}(y), \ldots, a_\nu^{(r_\nu)}(y)$ sont des fonctions réelles continues données, définies dans la région Ω_ν, où bien discontinues définies dans le disque D_ν $(\nu = 1, \ldots, p)$; 4) $f(y)$ est une fonction réelle, définie dans la somme de régions $\Omega = \Omega_1 + \ldots + \Omega_p$.

On admet les hypothèses suivantes.

I. Toute surface $S_\nu^{(\gamma)}$ est une surface de Liapounoff, soit fermée, soit ayant des bords fermés à $n-3$ dimensions.

II. L'angle $[s_\nu^{(j)}(y), s_\nu^{(j)}(\tilde{y})]$ entre les directions du même champ aux points arbitraires y, $\tilde{y}$ de tout domaine partiel $\Omega_\nu^{(\gamma)}$ du disque D_ν vérifie une inégalité

$$[s_\nu^{(j)}(y), s_\nu^{(j)}(\tilde{y})] < \text{const}\, |y - \tilde{y}|^{h_s} \tag{3}$$

$(\nu = 1, 2, \ldots, p;\ j = 1, 2, \ldots, r_\nu)$, h_s est une constante positive ne dépassant pas l'unité; les directions du champ $s_\nu^{(j)}(y)$ peuvent donc être discontinues aux surfaces $S_\nu^{(\gamma)}$.

III. Toute fonction $a_\nu^{(j)}(y)$ vérifie une condition de Hölder

$$|a_\nu^{(j)}(y) - a_\nu^{(j)}(\tilde{y})| < k_a |y - \tilde{y}|^{h_a} \tag{4}$$

dans tout domaine $\Omega_\nu^{(1)}, \ldots, \Omega_\nu^{(q_\nu)}$ *séparément*; k_a, h_a sont des constantes positives $(0 < h_a \leqslant 1)$; $|y - \tilde{y}|$ désigne la distance euclidienne des deux points y et $\tilde{y}$ du domaine $\Omega_\gamma^{(j)}$.

IV. La fonction $f(y)$ définie dans Ω, appartient à la classe $\mathfrak{H}_a^h$ relativement à toute region Ω_ν, c.à.d. vérifie les inégalités (voir [1])

$$|f(y)| < \frac{M_f}{|y - y_s)^a},$$

$$f(y) - f(\tilde{y})| < \frac{k_f |y - \tilde{y}|^h}{|y - y_s|^{a+h}} \tag{5}$$

y et $\dot{y}$ étant les deux points arbitraires, situés simultanément à l'intérieur des domaines $\Omega_\nu^{(1)}, \ldots, \Omega_1^{(q)}$; on a $|y - y_s| \leqslant |\tilde{y} - \tilde{y}_s|$; y_s désigne un point des surfaces S_y^ν le plus approché du point y; M_f, k_f, a, h sont des constantes positives données, on admet les inégalités

$$0 < a < 1; \quad a + h < 1 \tag{5'}$$
$$0 < h < \min(h_a, h_s).$$

2. Résolution du problème

Nous cherchons la solution du problème sous la forme d'un potentiel de simple couche

$$u(x) = \int_\Omega \frac{\varphi(z)\, dz}{|x - z|^{n-2}} \tag{6}$$

repandu sur l'ensemble de points $\Omega = \Omega_1 + \ldots + \Omega_p$; la densité $\varphi(z)$ est une fonction inconnue dans la région Ω, appartenant à la classe $\mathfrak{H}$ relativement à toute région Ω_ν.

En demandant que la fonction (6) vérifie la condition (2) en tout point $y \in \Omega$ et en s'appuyant sur les propriétés bien connues de la théorie du potentiel, on arrive à une équation intégrale *singulière*

$$-\lambda_n \varphi(y) + \int_{\Omega} \sum_{j=1}^{r_v} \frac{(n-2)a_v^{(j)}(y) \cos[\Theta_v^{(j)}(y, z)]}{|y-z|^{n-1}} \varphi(z)\, dz + f(y) = 0, \tag{7}$$

$(y \in \Omega_v)$ où $\lambda_n = \dfrac{(\sqrt{\pi})^n}{\Gamma(n/2)}$; $\Theta_v^{(j)}(y, z)$ est l'angle entre le vecteur $y - z$ et la direction $s_v^{(j)}(y)$ au point y; l'intégrale singulière a le sens de la valeur principale de Cauchy.

La théorie actuelle des équations intégrales singulières dans les régions à plusieurs dimensions est impuissante à résoudre l'équation (7) en toute généralité. Cependant, en s'appuyant sur les propriétés des fonctions de classe $\mathfrak{H}$, étudiée dans [1], et sur le théorème topologique connu de J. Schauder [2] sur le point invariant d'une transformation dans l'espace de Banach, nous démontrerons l'existence d'au moins une solution de l'équation (6), si les bornes supérieures des fonctions $a_v^{(j)}(y)$ et leur coefficient de Hölder k_a dans l'inégalité (4) sont suffisamment petits.

Nous démontrerons d'abord un théorème auxiliaire qui donne une extension de la propriété étudiée dans [1].

Théorème auxiliaire. *Si la fonction réelle $f(y)$, définie dans la région Ω_v sur le plan E^{n-1}, appartient à la classe $\mathfrak{H}_a^h$, donc vérifie les inégalités* (5), *si la fonction réelle $F(y, t)$ est définie par la formule* ($|y|$ *désigne la norme du point* $y \in E^{n-1}$)

$$F(y, t) = \frac{K(y', t)}{|y|^{n-1}}; \tag{8}$$

dans la région $[y \in E^{n-1} - \theta;\ t \in \Omega^*]$, où θ désigne l'origine des coordonnées dans le plan E^{n-1}, Ω^* est un domaine quelconque, $y' = y/|y|$ est une projection centrale du point y sur la surface ω d'une sphère dans le plan E^{n-1}, centrée en θ, de rayon unitaire, $K(y', t)$ est une fonction définie pour $[y' \in \omega,\ t \in \Omega^*]$, vérifiant la condition de Hölder

$$|K(y', t) - K(y'', \tilde{t})| < k_F^{-}[|y' - y''|^{h_\omega} + |t - \tilde{t}|^{h_1}] \tag{9}$$

et la condition intégrale

$$\int_{\omega} K(y', t)\, dy' = 0 \tag{10}$$

quelque soit $t \in \Omega^*$, alors l'intégrale singulière au sens de Cauchy

$$\Phi(y, t) = \int_{\omega_v} F(y - z, t) f(z)\, dz \tag{11}$$

est une fonction, définie dans la région $[y \in \Omega_v^*,\ t \in \Omega^*]$, vérifiant les inégalités

$$|\Phi(y, t)| < \frac{C M_F (M_f + k_f)}{|y - y_S|^a} \tag{12}$$

$$|\Phi(y, t) - \Phi(\tilde{y}, \tilde{t})| < \frac{C' (M_F + k_F)(M_f + k_f)[|y - \tilde{y}|^h + |t - \tilde{t}|^h]}{|y - y_S|^{a+h}} \tag{12'}$$

$(|y-y_S| \leqslant |\tilde{y}-\tilde{y}_S|)$ *C et C' sont des constantes positives indépendantes des fonctions F et f. On admet que*: $h < \min(h_\omega, h_1)$, $\alpha > 0$, $\alpha+h < 1$.

Démonstration. Les propriétés (12) et (12'), dans le cas $t = \tilde{t}$, résultent des considérations dans notre travail [1]. Il reste donc à prouver la propriété (12') pour le cas $y = \tilde{y}$. Dans ce but il suffit d'étudier l'intégrale

$$J_j(y,t) = \int\limits_{\Omega_\nu^{(j)}} F(y-z,t) f(z)\, dz \tag{13}$$

étendue à l'un des domaines partiels de la région Ω_ν et dans le cas $|t-\tilde{t}| \leqslant \frac{1}{2}|y-y_S|$. Admettons d'abord que $y \in \Omega_\nu^{(j)}$ et décomposons l'intégrale (13) en somme d'intégrales

$$J_j(y,t) = J_j^{A}(y,t) + J_h^{\Pi-A}(y,t) + J_j^{\Omega_\nu^{(j)}-\Pi}(y,t) \tag{14}$$

étendues: 1) à la sphère A, centrée en y, de rayon $|t-\tilde{t}|$, 2) à la portion $\Pi - A$ de la sphère Π, centrée en y, de rayon $|y-y_S|$, 3) à la partie extérieure $\Omega_\nu^{(j)}-\Pi$.

Nous aurons, de même que dans [1],

$$|J_j^{A}(y,t)| < \int\limits_{A} \frac{M_F\, k_f\, dz}{|y-z|^{n-1-h}\, |z-z_S|^{\alpha+h}} < \frac{\text{const}\, M_F k_f |t-\tilde{t}|^h}{|y-y_S|^{\alpha+h}} \tag{15}$$

et une limitation analogue pour le point $\tilde{t}$. Nous écrivons ensuite, d'après l'hypothése (9),

$$|J_j^{\Pi-A}(y,t) - J_j^{\Pi-A}(y,\tilde{t})| < k_F M_f \int\limits_{\Pi-A} \frac{|t-\tilde{t}|^{h_1}\, dz}{|y-z|^{n-1}\, |z-z_\Pi|^{\alpha}} = \tag{16}$$

$$= \text{const}\ k_F M_f \frac{|t-\tilde{t}|^{h_1}}{|y-y^\alpha|^S} \int\limits_{|t-\tilde{t}|/|y-y_S|} \frac{ds}{s(1-s)^\alpha} < \text{const}\ k_F M_f \frac{|t-\tilde{t}|^h}{|y-y_S|^\alpha}.$$

D'une façon analogique on obtient une limitation pour la différence concernant le troisième terme dans (14) et pour la différence $J_j(y,t) - J_j(y,\tilde{t})$, dans la décomposition (14), si $y \in \Omega_\nu - \Omega_\nu^{(j)}$ (voir [3]). En réunissant les résultats obtenus, on arrive à conclusion (12') de notre théorème.

Passons maintenant à l'étude de l'équation intégrale singulière (7) par la méthode topologique de J. Schauder, qui exige maintenant des considérations plus délicates, en vue des surfaces de discontinuité des fonctions du problème. Soit donc un espace A composé de toutes les fonctions réelles $\varphi(y)$, définies et continues dans la région *ouverte* Ω, vérifiant l'inégalité

$$\sup_{y \in \Omega} |y-y_S|^{\alpha+h} |\varphi(y)| < \infty, \tag{17}$$

α et h étant des constantes positives, figurant dans la propriété admise (4) de la fonction donnée $f(y)$. On admet la définition suivante de la norme du point $\varphi(y)$

$$\|\varphi(y)\| = \sup_{y \in \Omega} [|y-y_S|^{\alpha+h} |\varphi(y)|]. \tag{18}$$

En admettant les définitions évidentes des opérations linéaires et de la distance des deux points, on constate que l'espace Λ est linéaire, normé et métrique. En outre, on peut montrer sans peine qu'il est *complet*, il est donc un espace de Banach. Considérons maintenant dans cet espace un ensemble $\mathcal{E}$ de tous les points $\varphi(y)$ vérifiant les inégalités

$$
(19) \qquad \begin{cases} |y-y_S|_\alpha \, |\varphi(y)| \leqslant \varrho \\ |y-y_S|^{\varkappa+h} \, |\varphi(y)-\varphi(\tilde{y})| \leqslant \varkappa \, |y-y|^h \end{cases}
$$

ϱ et $\varkappa$ étant les deux constantes positives arbitrairement fixées, y et $\tilde{y}$ étant situés à l'intérieur du même domaine composant de Ω et $|y-y_S| \leqslant |\tilde{y}-\tilde{y}_S|$. Les fonctions $\varphi(y)$, composantes de l'ensemble ε, appartiennent donc à la classe $\mathfrak{H}_\nu^h$. L'ensemble $\mathcal{E}$ est évidemment fermé et convexe. Transformons maintenant l'ensemble $\mathcal{E}$ en faisant correspondre à tout point $\varphi(y)$ de cet ensemble un point $\psi(y)$ défini par la relation intégrale (voir (7))

$$
(20) \quad -\lambda_n \, \psi(\mathcal{X}) + \int\limits_{\Omega} \sum_{j=1}^{l} \frac{(n-2)\, a_\nu^{(j)}(y) \cos\, [\Theta_\nu^{(j)}(y,z)]}{|y-z|^{n-1}} \, \varphi(z)\, dz + f(y) = 0 .
$$

En vertu de nos hypothèses et du théorème auxiliaire, la fonction $\psi(y)$ est définie dans la région Ω et appartient à la même classe $\mathfrak{H}_\alpha^h$, relativement aux régions Ω_ν, que la fonction $f(y)$; notamment elle vérifie les inégalités

$$
(21) \quad \begin{cases} |y-y_S|^\alpha |\psi(y)| \leqslant \lambda_n^{-1} M_f + C_1 M_a(\varrho+\varkappa), \\ |y-y|^{\alpha+h} |\psi(y)-\psi(\tilde{y})| \leqslant [\lambda_n^{-1} k_f + C_2 (M_a+k_a)(\varrho+\varkappa)] \, |y-\tilde{y}|^h \end{cases}
$$

($y, \tilde{y} \in \Omega_\nu^{(j)}$), C_1 et C_2 sont des constantes positives, indépendantes des fonctions $a_\nu^{(j)}(y)$, $f(y)$, $\varphi(y)$, $M_a = \sup |a_\nu^{(j)}(y)|$. La fonction transformée $\psi(y)$ appartient donc à l'ensemble $\mathcal{E}$ sous la condition suffisante que les constantes du problème vérifient les inégalités

$$
(22) \qquad \begin{cases} \lambda_n^{-1} M_f + C_1 M_a(\varrho+\varkappa) \leqslant \varrho, \\ \lambda_n^{-1} k_f + C_2 (M_a+k_a)(\varrho+\varkappa) \leqslant \varkappa. \end{cases}
$$

Les constantes positives ϱ et $\varkappa$ étant jusqu'ici arbitraires, on peut montrer par un calcul élémentaire que leurs choix, pour satisfaire aux inégalités (22), est possible sous la condition nécessaire et suffisante que les constantes du problème M_a et k_a vérifient l'inégalité

$$
(23) \qquad (C_1+C_2)\, M_a + C_2\, k_a < 1.
$$

Dans ce cas l'ensemble $\mathcal{E}'$, transformé de l'ensemble $\mathcal{E}$ par la relation (20), fait partie de l'ensemble $\mathcal{E}$. On peut montrer d'une façon analogique à notre travail [3] que la transformation (20) est *continue* dans l'espace Λ et que l'ensemble transformé $\mathcal{E}'$ est *compact*. Nous en concluons, en vertu du théorème connu de J. Schauder (voir [2] et [3]), l'existence d'au moins une solution $\varphi^*(y) \in \mathfrak{H}_h^\alpha$ de l'équation (7), donc d'au moins une fonction harmonique (6) étant la solution du problème proposé, si la condition (23) est vérifiée.

On peut aussi résoudre l'équation (7) par la méthode des approximations successives. mais sous condition plus restrictive pour les constantes du problème que la condition (23).

INSTITUT MATHÉMATIQUE, ACADÉMIE POLONAISE DES SCIENCES
(INSTYTUT MATEMATYCZNY, PAN)

OUVRAGES CITÉS

[1] W. Pogorzelski, *Sur une classe des fonctions discontinues et une intégrale singulière dans l'espace*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **8** (1960), 445—451.

[2] J. Schauder, *Der Fixpunktsatz in Funktionalräumen*, Studia Math. (1930), 171—180.

[3] W. Pogorzelski, *Propriétés d'une intégrale singulière dans l'espace et leur application à un système d'équations intégrales singulières* [en russe]. Volume publié à l'occasion du 70^{me} anniversaire de M.N.I. Mouskhelichvili, Moscou et Gröningen, P. Noordhoffs, 1961, [trad. angl. publiée par Soc. for Ind. and Appl. Math., U.S.A.].

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 5, 1961

MATHEMATIK

H_0^A-Räume und ihre Anwendung auf allgemeine Eigenfunktionsentwicklungen

von

K. MAURIN

Vorgelegt von A. MOSTOWSKI den 14. März 1961

In den Abhandlungen [4], [5] hat der Verfasser den Begriff der Hilbert-Schmidtschen (H.-S.) Abbildungen eingeführt und für verschiedene Probleme der Analysis ausgenutzt. Es erwies sich, dass die H.-S.—Eigenschaft der Einbettungen für abstrakte Eigenfunktionsentwicklungen fundamental ist: es konnten auf diese Weise viele Spektralsätze unter einen Hut gebracht und neue Resultate gewonnen werden.

Die vorliegende Note ist eine Fortsetzung der Untersuchungen von [4], [5]. Dort wurde die H.-S.—Eigenschaft der Einbettung $H^{m+k}(\Omega_n) \to H^k(\Omega_n)$, $m > n/2$, $k \geqslant 0$ bewiesen. Die hier definierte Räume $H_0^A(\Omega_n)$ und ihre induktiven Limites $\mathcal{H}^A(\Omega) = \text{l.ind}\, H_0^A(\Omega^p)$, wo $\Omega^p \nearrow \Omega$, umfassen die seit einigen Jahren in der Theorie der Differentialoperatoren ständig benutzten Hilbertschen Räume $H_0^m(\Omega_n)$. Es wird die H.-S.—Eigenschaft der Einbettungen $H_0^A(\Omega_n) \to H_0^B(\Omega_n)$ bewiesen, was eine Anwendung eines in [4] bewiesenen Satzes erlaubt. Auf diese Weise konnte ein allgemeiner Eigenfunktionsentwicklungssatz in $L^2(\Omega)$ gewonnen werden, der alle ähnliche Resultate von Berezanskiĭ, Browder, Gelfand-Kostiučenko, Gårding, Maurin u.a. verallgemeinert oder verschärft.

1. H.-S.—Eigenschaft der Einbettung $H \to L^2(\Omega_n)$

DEFINITION. Es seien H, F separable Hilbertsche Räume. Die lineare Abbildung $L : H \to F$ ist vom H.-S.—Typus falls

$$H \ni \varphi \to L\varphi \overset{\text{df}}{=} \sum_{i=1}^{\infty} (\varphi, e_i)_H f_i \doteq \sum_{i=1}^{\infty} (\varphi, e_i)_H Le_i \in F, \tag{1}$$

wobei

$$\sum \|f_i\|_F^2 = \sum \|Le_i\|_F^2 < \infty \text{ für ein v.o.S.} (e_i) \tag{2}$$

in H gilt.

SATZ 1. *Es sei H ein Hilbertscher Raum mit dem Skalarprodukt $(\cdot, \cdot)$, wobei $H \subset L^2(\Omega)$. Ω — ein beliebiger Bereich des n — dimensionalen Euklidischen Raumes,*

oder eine differenzierbare Mannigfaltigkeit. Es sei weiter für jedes $x \in \Omega$ $H \ni \varphi \to \to \varphi(x) \in \mathcal{C}^1$ *ein stetiger Funktional auf* H. *Es gibt ein solches* $b_x \in H$, *dass*

$$\varphi(x) = (\varphi, b_x). \tag{3}$$

Falls

$$\Omega \ni x \to \|b_x\|^2 \in E^1 \tag{4}$$

intergrierbar ist, dann ist die Einbettung $H \ni \varphi \to \varphi \in L^2(\Omega)$ *vom H.-S.—Typus.*

Beweis. Aus der Parsevalschen Gleichung und aus (3) folgt

$$\| b_x \|^2 = \sum |(e_i, b_x)|^2 = \sum | e_i(x) |^2. \tag{5}$$

Wegen (4) dürfen wir die rechte Seite von (5) gliedweise integrieren:

$$\infty > \int_\Omega \| b_x \|^2 \, dx = \sum_{i=1}^{\infty} \int_\Omega | e_i(x) |^2 \, dx = \sum_{i=1}^{\infty} \|e_i\|^2 \in L^2.$$

Die Einbettung $H \to L^2(\Omega)$ ist also vom H.-S.—Typus.

Aus diesem Satze folgen sofort für beschränktes Ω wichtige Korollare.

KOROLLAR 1. (vgl. [4]). *Falls das Mass* $|\Omega_n|$ *von* Ω_n *endlich ist und* Ω_n *den Sobolevschen Bedingungen genügt, dann ist die Einbettung* $H^m(\Omega_n) \to L^2(\Omega_n)$ *für* $m \geqslant [n/2] + 1$ *vom H.-S.—Typus.*

Beweis. Wegen der Ungleichung von Sobolev

$$|\varphi(x)| \leqslant c(\Omega_n) \|\varphi\|_m, \quad \varphi \in H^m(\Omega_n)$$

sind die Voraussetzungen des Satzes 1 erfüllt.

KOROLLAR 2. *Es sei* $H_0^1(\Omega_n)$ *die Vervollständigung der Menge* $C_0^\infty(\Omega_n)$ *in der Norm*

$$\|\varphi\|_1^2 \overset{\mathrm{df}}{=} \int_{\Omega_n} \left| \frac{\partial^n \varphi}{\partial x_1 \ldots \partial x_n} \right|^2 dx \tag{6}$$

(vgl. den folgenden Paragraph).

Dann ist die Einbettung $H_0^1(\Omega_n) \to L^2(\Omega_n)$ *vom H.-S.—Typus.*

Beweis. Für jedes $\varphi \in C_0^\infty(\Omega_n)$ gilt die elementare Ungleichung

$$|\varphi(x)| \leqslant c(\Omega) \|\varphi\|_1 \tag{7}$$

was als Beweis dienen kann.

Bemerkung. Aus Korollar 2 und aus einem Spektralsatz des Verfassers [4] folgt ein Verschärfungssatz von Gelfand-Kostiučenko [1] über die Struktur der verallgemeinerten Eigenfunktionen (vgl. Par 3).

2. Die Räume $H_0^A(\Omega_n)$ und ihre Einbettungen

Jetzt führen wir eine natürliche Verallgemeinerung der Räume $H_0^m(\Omega_n)$ ein die bekanntlich eine Vervollständigung der Menge $C_0^\infty(\Omega_n)$ (beliebig oft differenzierbaren Funktionen mit kompakten Trägern in Ω_n) in der $\| \; \|_m$ — Norm ist:

$$(u, v)_m = \sum_{|\alpha| \leqslant m} (D^\alpha u, D^\alpha v)_0, \quad (u, v)_0 = \int_{\Omega_n} uv\, dx$$

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^1 \dots \partial x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \dots + \alpha_n,$$

$$\alpha = (\alpha_1, \dots, \alpha_n).$$

Es sei N_+^n die Menge der nichtnegativen Gitterpunkte in E^n. Es sei A eine endliche Untermenge von N_+^n, Ω_n ein beschränkter Bereich in E^n

(8) $$(\varphi, \psi)_A \stackrel{\text{df}}{=} \sum_{\alpha \in A} (D^\alpha \varphi, D^\alpha \psi)_0, \quad \|\varphi\|_A \stackrel{\text{df}}{=} (\varphi, \varphi)_A.$$

Wir führen — nach S. Łojasiewicz [3] — die konvexe Menge $C(A) \subset E_+^n$ folgendermassen ein

$$C(A) = \left\{x \in E_+^n: \quad 0 \leqslant x < \sum \lambda_i(x) \alpha^i, \alpha^i \in A\right\},$$

wobei

$$\lambda_i(x) \geqslant 0, \quad \sum \lambda_i(x) = 1.$$

Die Ungleichung $x < y$ bedeutet, dass $x_\nu < y_\nu$, $\nu = 1, \dots, n$ wo $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$. Wir nehmen weiter an, dass

$$C(\{0\}) \stackrel{\text{df}}{=} \{0\}.$$

Es bedeute schliesslich $\mathbf{1} = \underbrace{(1, \dots, 1)}_{n}$. Dann gelten die folgenden Ungleichungen von Łojasiewicz:

Es seien A, B endliche Untermengen von N_+^n, wobei $\frac{1}{2}\mathbf{1} + B = \{\beta + \frac{1}{2}\mathbf{1} : \beta \in B\} \subset \subset C(A)$, dann gibt es eine solche (von x und φ unabhängige) Konstante $K = K(A, \Omega_n)$, dass

(9) $$|D^\beta \varphi(x)| \leqslant K(A, \Omega_n) \|\varphi\|_A, \quad \text{für} \quad \beta \in B, \quad \varphi \in C_0(\Omega_n)$$

(10) $$\|\varphi\|_{A_0} \leqslant K(A, \Omega_n) \|\varphi\|_A \quad \text{für jedes} \quad A_0 \subset C(A) \cap N_+^n, \quad \varphi \in C_0^\infty(\Omega_n).$$

Im folgenden wird immer vorausgesetzt, dass $H_0^A(\Omega_n) \subset L^2(\Omega_n)$ wegen (10) ist hinreichend (und notwending), dass $0 \in C(A)$.

Es gilt der folgende

Satz 2. *Es seien A, B endliche Untermengen von N_+^n, wobei $B - \frac{1}{2}\mathbf{1} \subset C(A)$. Dann ist die Einbettung $H_0^A(\Omega_n) \to H_0^B(\Omega_n)$ vom H.-S.—Typus.*

Beweis. Da $C_0^\infty(\Omega_n)$ dicht in $H_0^A(\Omega_n)$ ist, und wegen (9), haben wir die folgende Ungleichung:

(11) $$|D^\beta u(x)| \leqslant K(A, \Omega_n) \|u\|_A \quad \text{für } u \in H_0^A(\Omega_n), \ \beta \in B.$$

Diese Ungleichung bedeutet aber, dass die Abbildungen

$$H_0^A(\Omega_n) \ni u \to D^\beta u(x) \in C^1, \quad x \in \Omega_n$$

stetige lineare Funktionale auf H_0^A sind. Es gibt also ein solches $b_x^\beta \in H_0^A(\Omega_n)$, dass

$$D^{\beta|}u(x) = (\varphi, b_x^\beta)_A, \quad x \in \Omega_n, \quad \beta \in B.$$

Daraus und aus der Parsevalschen Gleichung und (11) folgt

$$\|b_x^\beta\|_A^2 = \sum_i |(e_i, b_x^\beta)_A|^2 = \sum_i |D^\beta e_i(x)|^2 < K^2, \tag{12}$$

für jeden v. o. $S.(e_i)$ in $H_0^A(\Omega_n)$.

Da das Mass $|\Omega_n|$ von Ω_n endlich ist, ist $x \to \|b_x^\beta\|_A^2$ intergrierbar auf Ω_n. Man darf also die rechte Seite von (12) unter dem Summenzeichen integrieren:

$$\infty > \sum_{\beta \in B} \int_{\Omega_n} \|b_x^\beta\|_A^2 \, dx = \sum_{\beta \in B}' \sum_i \|D^\beta e_i\|_0^2 = \sum_i \sum_{\beta \in B} \|D^\beta e_i\|_0^2 = \sum_i \|e_i\|_B^2.$$

Die Einbettung $H_0^A(\Omega_n) \to H_0^B(\Omega_n)$ ist also vom H.-S.-Typus.

Korollar 3. *Falls $\frac{1}{2}\mathbf{1} \in C(A)$, dann hat die Einbettung*

$$H_0^A(\Omega_n) \to L^2(\Omega_n) \tag{13}$$

die H.-S.-Eigenschaft.

Beispiele:

1. Wenn man $A = \{\mathbf{1}\}$ nimmt, bekommt man wieder Korollar 1.
2. $A = \{a^k = (0, \ldots, \underset{k}{\underbrace{n}}, \ldots, 0)\}$.

Korollar 4. *Es sei $A_0 \subset C(A)$, $A_1 - \frac{1}{2}\mathbf{1} \subset C(A_0)$. Dann ist die Einbettung* (14) *H*

$$H_0^A(\Omega_n) \to H_0^{A_1}(\Omega_n) \tag{14}$$

vom H.-S.-Typus.

Beweis. Man kann (14) folgendermassen darstellen $H_0^A \to H_0^{A_0} \to H_0^{A_1}$. Aber die erste Abbildung ist wegen (10) stetig — die zweite nach Satz 2 ist vom H.-S.-Typus. Aber, wie in [4] bewiesen, ist Produkt einer stetigen und einer H.-S.-Abbildung wieder vom H.-S.-Typus.

Bemerkung. Da das Produkt zweier H.-S.-Abbildungen (sogar) nuklear ist [4], kann man aus Satz 2 verschiedene Sätze über Nuklearität der Einbettungen gewinnen:

Satz 3. *Falls $A_1 - \frac{1}{2}\mathbf{1} \subset C(A_0)$, $A_0 - \frac{1}{2}\mathbf{1} \subset C(A)$, ist die Einbettung $H_0^A(\Omega_n) \to$ $\to H_0^{A_1}(\Omega_n)$ nuklear.*

3. Eine Anwendung auf allgemeine Eigenfunktionsentwicklungen

In den Abhandlungen [4], [5] wurde die fundamentale Rolle der H.-S.-Einbettungen für allgemeine Eigengunktionsentwicklungen erkannt. Wenn man den dort bewiesenen Hauptsatz mit hier gewonnenen Ergebnissen verbindet, gewinnt man eine Reihe von allgemeinen Spektralsätzen in $L^2(\Omega)$ oder anderen Hilbertschen Räumen $H(b)$ (wegen Definition von $H(b)$ vgl. [4]). Hier sei nur die folgende Aussage formuliert:

SATZ 4. *Es sei* (L_ν) *ein vertauschbares System selbstadjungierter Differentialoperatoren in* $L^2(\Omega)$, *wo* Ω *ein beliebiger Bereich in* E^n *ist. Es sei* A *eine solche endliche Untermenge von* N_+^n, *dass* $\frac{1}{2}\mathbf{1} \subset C(A)$. *Dann gibt es ein vollständiges System gemeinsamer verallgemeinerten Eigenelemente* $e_k(\lambda)$ *von* (L_ν), *wobei* $e_k(\lambda) \in \mathcal{H}^{-A}(\Omega) \overset{\mathrm{df}}{=} (\mathcal{H}^A(\Omega))'$.

$\mathcal{H}^A(\Omega) = \mathrm{l.\ ind}\ H_0^A(\Omega^p)$, *wo* Ω^p — *beschränkt sind und* $\Omega^p \nearrow \Omega$ *für* $p \to \infty$. *Die* $e_k(\lambda)$ *sind also stetige Funktionale auf* $\mathcal{H}^A(\Omega)$, *also — Distributionen von Ordnung* $\subset A$ (*vgl.* [3]).

KOROLLAR 5. *Falls man* $A = \{\mathbf{1}\}$ *nimmt, bekommt man eine Verschärfung eines Satzes von Gelfand-Kostiučenko.*

KOROLLAR 6. *Nimmt man* $A = \{\alpha \in N_+^n,\ |\alpha| \leqslant [n/2]+1\}$, *so bekommt man einen Satz des Verfassers* (Korollar 5 in [5]).

KOROLLAR 7. *Nimmt man* $A = \{\alpha : |\alpha| = n$, *so bekommt man einen Satz von L. Maurin und dem Verfasser* [2].

MATHEMATISCHES INSTITUT, POLNISCHE AKADEMIE DER WISSENSCHAFTEN
(INSTYTUT MATEMATYCZNY, PAN)

SCHRIFTTUM

[1] I. M. Gelfand, A. G. Kostiučenko, *Über Entwicklungen nach Eigenfunktionen der Differential- und anderer Operatoren* [russisch], Dokl, A.N.S.S.S.R., **103** (1955), 349—352.

[2] K. Maurin, L. Maurin, *Nuklearität gewisser Rellich-Sobolevschen Einbettungen, Anwendung auf Spektraltheorie der Differentialoperatoren*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **8** (1960), 621—624.

[3] S. Łojasiewicz, *Sur la fixation des variables dans une distribution*, Studia Math., **17** (1958), 1—64.

[4] K. Maurin, *Mappings of Hilbert-Schmidt Type, their applications to eigenfunction-expansions and elliptic boundary problems*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys. **9** (1961), 7—11.

[5] —, *Abbildungen vom Hilbert-Schmidtschen Typus. Ihre Anwendungen auf Eigenfunktionsentwicklungen und elliptischen Randwertaufgaben*, Math. Scand. [im Druk].

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 5, 1961

MATHEMATICS

A Compact Space of Models of First Order Theories

by

A. EHRENFEUCHT and A. MOSTOWSKI

Presented by A. MOSTOWSKI on March 22, 1961

Let T be a consistent first order theory whose primitive notions are exclusively relations and which does not have finite models. We assume that the identity predicate occurs among the primitive notions of T. The aim of this note is to prove the

THEOREM. *There exists a family* $\mathfrak{S}$ *of models of* T *satisfying* (1)—(4) *below*:

(1) *The domain of each* M *in* $\mathfrak{S}$ *is the set of non negative integers*;

(2) $\mathfrak{S}$ *is a bicompact topological space*;

(3) *for every formula* Φ *of* T *and every assignment* f *of integers to the free variables of* Φ *the set*

$$\{M : \Phi \text{ is satisfied in } M \text{ by the assignment } f\}$$

is open and closed in $\mathfrak{S}$;

4) *for every denumerable model of* T *there is an isomorphic model in* $\mathfrak{S}$.

The introduction of topology to the set of denumerable models of a theory is of course no novelty (cf. [1]—[3]).*) Under these topologies models do not, i . general, form a compact space unless T satisfies certain additional conditions (cf. [4]). The authors believe that compactness of $\mathfrak{S}$ together with the continuity property (3) and the universality property (4) may find some applications.

The construction of $\mathfrak{S}$ will be carried out in two steps. In the first step we construct an auxiliary space $\mathfrak{P}$ whose elements are not models but relational systems in which the identity symbol is interpreted as a relation in general different from the relation $=$. In the second step we map $\mathfrak{P}$ onto a new space $\mathfrak{S}$ which has all the properties required in the Theorem. **)

1. We first extend T to an auxiliary theory T^0; this new theory is obtained from T by adding to it all possible "Skolem functors" for formulas of T.

---

*) The first step towards the use of general topology in the theory of models was made by Blake [1] in a proof of Gödel completeness theorem. His topological proof of this theorem was presented and further developed in [2]. Rasiowa and Sikorski [3] approached the problem from a different and very fruitful point of view.

**) The first part of the proof was found by the second author who also formulated the problem discussed in this note. The second part of the proof is the work of the first author.

T^0 is defined as the union of theories T_n where T_0 has the same language as the given theory T and as axioms only the axioms of identity. T_{r+1} is obtained from T_r by the following process.

For every formula Φ of T_r which is not a formula of T_{r-1} and for an arbitrary integer k we add to T_r a new functor (i.e., a symbol for a function) $\mathfrak{f}_{\Phi,k}$ with as many arguments as there are free variables in Φ different from the variable v_k. Relational symbols remain unchanged. Axioms of T_{r+1} are those of T_r together with all axioms of the form

$$(v_k, v_{i_1}, \ldots, v_{i_p})\,[\Phi\,(v_k, v_{i_1}, \ldots, v_{i_p}) \supset \Phi\,(\mathfrak{f}_{\Phi,k}\,(v_{i_1}, \ldots, v_{i_p}),\ v_{i_1}, \ldots, v_{i_p})],$$

where Φ is a formula of T_r which is not a formula of T_{r-1} and $v_k, v_{i_1}, \ldots, v_{i_p}$ are all of its free variables.

We next define a Skolem resolvent Φ^{Sk} of a formula Φ of T^0. If Φ has no quantifiers then Φ^{Sk} is Φ. If Φ is the formula $(Ev_k)\,\Psi$ and $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$ are all of its free variables then Φ^{Sk} is the formula $\mathrm{Sb}\,(v_k/\mathfrak{f}_{\Psi^{\mathrm{Sk}},k}\,(v_{i_1}, \ldots, v_{i_p}))\,\Psi^{\mathrm{Sk}}$ where Sb denotes the operation of substitution. Finally $(|\Phi_1\,\Phi_2)^{\mathrm{Sk}}$ is $|\Phi_1^{\mathrm{Sk}}\,\Phi_2^{\mathrm{Sk}}$. *)

Φ and Φ^{Sk} have the same free variables.

1.1. *$\Phi \equiv \Phi^{\mathrm{Sk}}$ is provable in T^0 for every formula Φ of T^0.*

2. The space $\mathfrak{P}$ of pseudo models. Let $\mathcal{E}$ be the set of terms of T^0, i.e. the smallest set which contains the variables and has the property that whenever $\mathfrak{f}$ is a functor of T^0 with, say, k arguments and $t_1, \ldots, t_k$ are in the set, then so is $\mathfrak{f}\,(t_1, \ldots, t_k)$.

Let I be set. A family of relations and functions indexed by the predicates and functors of T^0 is called a pseudo model of T^0 over I if this family satisfies the following conditions: Every relation from the family has the field I and as many arguments as its index; every function from the family has the domain I and the range contained in I and the number of its arguments is the same as the number of the arguments of the functor which serves as its index.

$\mathcal{E}$-pseudo models are pseudo models over $\mathcal{E}$ such that for every functor $\mathfrak{f}$ of T^0 the function with the index $\mathfrak{f}$ coincides with the function $F_{\mathfrak{f}}\,(t_1, \ldots, t_k) = \mathfrak{f}\,(t_1, \ldots, t_k)$ where k is the number of arguments of $\mathfrak{f}$.

We denote by $[\Phi]$, where Φ is an open formula of T^0, the set of $\mathcal{E}$-pseudo models P such that Φ is satisfied in P by the assignment $f: v_k \to v_k\ (k = 0, 1, 2, \ldots)$. Note that $v_k \in \mathcal{E}$ and thus f asigns an element of the domain of P to each variable.

2.1. *The family of $\mathcal{E}$-pseudo models is a compact separable Hausdorff space when sets $[\Phi]$ are taken as the open basis.*

Proof that Hausdorff axioms are satisfied is obvious. Separability follows from the denumerability of formulas.

Compactness is proved as follows: Let $\bigcap_{j \le n} [\Phi_j] \neq 0$ for $n = 0, 1, 2, \ldots$. Consider the Boolean algebra B whose elements are sets $[\Phi]$, where Φ runs over

*) $|$ is the Sheffer's stroke. Our formula says of course that the formation of resolvents is distributive over the operations of the propositional calculus.

open formulas of T^0. Sets $[\Phi]$, for which there exists an n such that $[\Phi] \supseteq \bigcap_{j\leq n} [\Phi_j]$, form a non trivial filter in B. Let $\mathcal{F}$ be its extension to a prime filter. If r is a relational symbol of T^0 and k is the number of its arguments then we define a relation R_r in $\mathcal{E}$ by the equivalence

$$R_r(t_1, \ldots, t_k) \equiv [r(t_1, \ldots, t_k)] \varepsilon \mathcal{F}.$$

We denote by P_0 the $\mathcal{E}$-pseudo model thus obtained.

We can now show by an easy induction that for every open formula Φ with the free variables $v_{i_1}, \ldots, v_{i_k}$ the following conditions are equivalent:

$$[\Phi] \varepsilon \mathcal{F}$$

Φ is satisfied in P_0 by the assignment f: $v_i \to v_i$.
Hence $P_0 \varepsilon \bigcap_{j<\infty} [\Phi_j]$ which concludes the proof of 2.1.

2.2. *The family $\mathfrak{P}$ of $\mathcal{E}$-pseudo models in which are true all axioms of T^0 as well as all formulas Φ^{Sk} where Φ is an axiom of T is closed in the space of all $\mathcal{E}$-pseudo models.*

This follows at once from the fact that axioms of T^0 and formulas Φ^{Sk} are open.

2.3. *$\mathfrak{P}$ considered as a topological space with topology induced by that of the space of all pseudo models is a compact separable space and has the properties:*

(3′) *for every formula Φ of T^0 and every assignment f of terms to its free variables the set*

$\{P : P \varepsilon \mathfrak{P}$ and Φ is satisfied in P by the assignment $f\}$ is open and closed in $\mathfrak{P}$;

(4′) *for every pseudo model Q over an at most denumerable set I in which all the axioms of T and T^0 are true there is an $\mathcal{E}$-pseudo model P in $\mathfrak{P}$ such that Q reduced modulo the equivalence relation E_Q which interprets $=$ in Q is isomorphic with P reduced modulo E_P.*

Proof. $\mathfrak{P}$ is compact as a closed subspace of a compact space.

(3′) Because of 1.1 it is sufficient to consider only the case when Φ is an open formula. If its free variables are $v_{i_1}, \ldots, v_{i_k}$ and f assigns a term t_i to v_i $(i = 0, 1, 2, \ldots)$, then the conditions: Φ is satisfied in P by f and $P \varepsilon [\Phi(t_{i_1}, \ldots, t_{i_k})]$ are equivalent, which proves (3′).

(4′) Let Q be a pseudo model over an at most denumerable set I and assume that all the axioms of T and of T^0 are true in Q. Let f be any mapping of the variables onto I. We extend f to a mapping of $\mathcal{E}$ onto I by putting $f(\mathfrak{f}(t_1, \ldots, t_k)) = F_{\mathfrak{f}}(f(t_1), \ldots, f(t_k))$ for arbitrary $t_1, \ldots, t_k$ in $\mathcal{E}$ and for an arbitrary functor $\mathfrak{f}$ of T^0 with k arguments. If S_r is the interpretation of a relational symbol r of T in Q then we define a relation R_r in $\mathcal{E}$ by the equivalence $R_r(t_1, \ldots, t_k) \equiv S_r(f(t_1), \ldots, f(t_k))$ for arbitrary $t_1, \ldots, t_k$ in $\mathcal{E}$. The $\mathcal{E}$-pseudo model determined by the relations R_r satisfies the condition (4′).

2.4. *If P is in $\mathfrak{P}$ then the number of equivalence classes of $\mathcal{E}$ under E_P is infinite.*

Proof. Otherwise T would have a finite model obtained from P by reducing it modulo E_P.

3. Proof of the Theorem. For every P in $\mathfrak{P}$ we define a mapping φ_P of $\mathcal{E}$ onto integers. Let

$$t_0, t_1, \ldots$$

be a sequence consisting of all elements of $\mathcal{E}$ and put $\varphi_P(t_0) = 0$,

$$\varphi_P(t_{n+1}) = \begin{cases} \varphi_P(t_n)+1 & \text{if } t_{n+1} \text{ non-}E_P\, t_j \text{ for } j=0,1,\ldots,n, \\ \varphi_P(t_j) & \text{if } j = \min_{i\leqslant n}[t_{n+1}\, E_P\, t_i]. \end{cases}$$

In view of 2.4 φ_P maps $\mathcal{E}$ onto integers.

From the definition of φ_P we immediately obtain the following three lemmas:

3.1. $\varphi_P(t_n) = \varphi_P(t_m) \equiv t_n\, E_P\, t_m$;

3.2. $\varphi_P(t_n) \leqslant n$;

3.3. *The necessary and sufficient condition for the equation* $\varphi_P(t_n) = k$ $(0 \leqslant k \leqslant n)$ *to hold is that there exist* $k+1$ *integers* $a_0, \ldots, a_k$ *such that* (i) $0 = a_0 < a_1 < \ldots < a_k$; (ii) $t_n\, E_P\, t_{a_k}$; (iii) t_{a_i} *non* $E_P\, t_{a_j}$, *for* $i \neq j$; (iv) *for every* $b \leqslant a_k$ *there is a* $p \leqslant k$ *such that* $t_b\, E_P\, t_{a_p}$.

3.4. *For arbitrary integers* k, n $(k \leqslant n)$ *there is a formula* $\Omega_{n,k}$ *of* T^0 *such that for every* P *in* $\mathfrak{P}$ *the conditions*

$$\varphi_P(t_n) = k \text{ and } P \,\varepsilon\, [\Omega_{n,k}]$$

are equivalent.

Proof. In view of 3.3 it is sufficient to take as $\Omega_{n,k}$ the disjunction of formulas

$$(t_n = t_{a_k}) \mathbin{\&} \bigwedge_{0\leqslant i<j\leqslant k} \sim (t_{a_i} = t_{a_j}) \mathbin{\&}$$

$$\bigwedge_{0\leqslant b\leqslant a_k} \bigvee_{0\leqslant p\leqslant k} (t_b = t_{a_p})$$

extended over sequences satisfying (i) of 3.3.

Let R_r be the relation which interprets in P the relational symbol r of T. Define a relation Q_r between integers as follows (k is the number of arguments of r)

$$Q_r(n_1, \ldots, n_k) \equiv \{\textit{there are } m_1, \ldots, m_k \textit{ such that}$$

$$(\varphi_P(t_{m_1}) = n_1) \mathbin{\&} \ldots \mathbin{\&} (\varphi_P(t_{m_k}) = n_k) \mathbin{\&} R_r(t_{m_1}, \ldots, t_{m_k})\}.$$

From 3.1 we obtain

3.5. *If* r *is the symbol* $=$ *then* Q_r *is the relation of identity.*

Let us denote by $\mu(P)$ the family of relations Q_r where r runs over the relational symbols of T.

3.6. $\mu(P)$ *is isomorphic to* P/E_P *and the isomorphic mapping is given by* $t/E_P \to \varphi_P(t)$ *for* $t \,\varepsilon\, \mathcal{E}$.

Proof. A relational symbol r of T is interpreted in P/E_P as the relation R'_r which holds between congruence classes C_i mod E_P if and only if there are t_{m_i} in C_i $(i = 1, 2, \ldots, k)$ such that $R_r(t_{m_1}, \ldots, t_{m_k})$. If we let correspond to C_i the integer

$\varphi_P(C_i) = \varphi_P(t_{m_i})$ which is independent of the particular element t_{m_i} chosen from C_i we obtain

$$R'_r(C_1, \ldots, C_k) \equiv Q_r(\varphi_P(C_1), \ldots, \varphi_P(C_k)).$$

Since the mapping $C \to \varphi_P(C)$ is one to one we obtain 3.6.

3.7. *If P is in $\mathfrak{P}$ then $\mu(P)$ is a model of T.*

Proof. If Φ is an axiom of T, then Φ^{Sk} is true in P and hence, by 1.1, Φ is true in P hence also in P/E_P and finally by 3.6 in $\mu(P)$.

3.8. *Any denumerable model of T is isomorphic to a model $\mu(P)$ with P in $\mathfrak{P}$.*

Proof. The lemma follows from 2.3 (4′) and the remark that if Q is a model, then E_Q is the identity relation.

3.9. *Conditions*

$$t_{i_1}, \ldots, t_{i_k} \textit{ satisfy } \Phi^{\mathrm{Sk}} \textit{ in } P$$

$$\varphi_P(t_{i_1}), \ldots, \varphi_P(t_{i_k}) \textit{ satisfy } \Phi \textit{ in } \mu(P)$$

are equivalent for any formula Φ of T with exactly k free variables and any terms $t_{i_1}, \ldots, t_{i_k}$.

Proof follows at once from 3.6 and 1.1.

Let now $\mathfrak{S}$ be the set of all $\mu(P)$ where P runs over $\mathfrak{P}$ and let a topology be introduced in $\mathfrak{S}$ by taking sets

$$W_{\Phi, f} = \{M : \Phi \textit{ is satisfied in } M \textit{ by the assignment } f\}$$

as neighbourhoods in $\mathfrak{S}$. Every neighbourhood is thus determined by a formula Φ and an assignment f of integers to its free variables.

$\mathfrak{S}$ obviously satisfies condition (1) of the Theorem. By 3.8 it satisfies condition (4). From the definition of the neighbourhoods it is obvious that it satisfies condition (3). Finally $\mathfrak{S}$ is a separable Hausdroff space. Thus it remains to show that $\mathfrak{S}$ is compact. Since a continuous image of a compact space is itself compact it will be sufficient to show that the mapping is continuous.

Let $\mu(P_0) \,\varepsilon\, W_{\Phi, f}$ and let the integers correlated (via f) to the free variables $v_{i_1}, \ldots, v_{i_k}$ of Φ be $n_1, \ldots, n_k$. Denote by $t_{j_1}, \ldots, t_{j_k}$ terms such that $\varphi_{P_0}(t_{j_s}) = n_s$ for $s = 1, \ldots, k$. Hence $P_0 \,\varepsilon\, [\Omega_{j_s, n_s}]$ for $s = 1, 2, \ldots, k$ and, in view of 3.9, P_0 belongs to $[\Phi^{\mathrm{Sk}}(t_{j_1}, \ldots, t_{j_k})]$. Thus, $\bigcap_{s \leq k} [\Omega_{j_s, n_s}] \cap [\Phi^{\mathrm{Sk}}(t_{j_1}, \ldots, t_{j_k})]$ is a neighbourhood U of P_0 in $\mathfrak{P}$ such that $\mu(P) \,\varepsilon\, W_{\Phi, f}$ for every P in U.

REFERENCES

[1] A. Blake, *Canonical expressions in Boolean algebra*, The University of Chicago Libraries, Chicago, 1938.

[2] A. Mostowski, *Logika matematyczna*. Monogr. Matem., **18** (1948) Warszawa—Wrocław.

[3] H. Rasiowa and R. Sikorski, *A proof of the completeness theorem of Gödel*. Fundam. Mathem., 37 (1950), 193—200.

[4] R. Sikorski, *A topological characterization of open theories*, Bull. Acad. Polon. Sci., Sér. Sci. math., astr. et phys. **9** (1961),

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 5, 1961

MATHEMATIK

Einige allgemeine Lückenumkehrsätze für permanente Toeplitzsche Limitierungsverfahren

von

W. KOŁODZIEJ

Vorgelegt von S. MAZUR am 27. März 1961

Gegeben sei ein permanentes Toeplitzsches Limitierungsverfahren $A = (a_{k,n})$ $(k, n = 0, 1, ...)$ vom Wirkfelde A^* und eine wachsende Folge $\{n_\nu\}$ der ganzen nichtnegativen Zahlen. Folgender Satz heisst *der Lückenumkehrsatz für das Verfahren A*:

Aus $t_n = 0$ *für* $n < n_0$, $t_n = s_\nu$ *für* $n_\nu \leqslant n < n_{\nu+1}$ $(\nu = 0, 1, ...)$ *und* $\{t_n\} \in A^*$, *folgt die Konvergenz der Folge* $\{t_n\}$.

Die Folge $\{n_\nu\}$ heisst dann *die Lückenfolge für das Verfahren A*.

Ist T eine gegebene Menge von Zahlenfolgen und wird im obigen Satze $\{t_n\} \in TA^*$ statt $\{t_n\} \in A^*$ gesetzt, so spricht man vom *Lückenumkehrsatze in der Klasse der Folgen* $\{t_n\} \in TA^*$.

Eine Funktion $f(n)$ $(n = 0, 1, ...)$ heisst *Lückenfunktion für das Verfahren A*, wenn jede wachsende, die Bedingung $n_{\nu+1} > f(n_\nu)$ $(\nu = 0, 1, ...)$ erfüllende Folge $\{n_\nu\}$ der ganzen, nichtnegativen Zahlen, die Lückenfolge für das Verfahren A ist.

In dieser Note werden einige, die Existenz der Lückenfolgen für permanente Limitierungsverfahren betreffende Sätze angegeben. Beweise dieser Sätze werden später veröffentlicht werden.

SATZ 1. *Ist das Verfahren A zeilenfinit, so besitzt es immer eine Lückenfunktion.*

Bemerkung 1. Im Satz 1 genügt es vorauszusetzen, dass $A^* \subset B^*$, wo B zeilenfinit und permanent ist.

Bemerkung 2. Jedes Verfahren A besitzt eine Lückenfunktion in der Klasse aller beschränkten A-limitierbaren Folgen.

Bemerkung 3. Ist das Verfahren A lückenperfekt [4], so besitzt es immer eine Lückenfunktion.

SATZ 2. *Das Verfahren* $A = (a_{k,n})$ *erfülle folgende Bedingungen*:

(α) $a_{0,n} \neq 0$ *für* $n \geqslant N_0$,

(β) *für jedes k ist* $\left\{\frac{a_{k,n}}{a_{0,n}}\right\}$ *eine Folge von beschränkter Variation, das heisst*:

$$\sum_{n=N_0}^{\infty}\left|\frac{a_{k,n}}{a_{0,n}}-\frac{a_{k,n+1}}{a_{0,n+1}}\right|<\infty .$$

Dann gibt es ein zeilenfinites, permanentes Verfahren B, für welches $A^* \subset B^*$.

Bemerkung 4. Gilt $a_{k,n} \neq 0$ für $n \geqslant N_k$ und ist $\left\{\frac{a_{k-1,n}}{a_{k,n}}\right\}$ für jedes k eine Folge von beschränkter Variation, so sind die Bedingungen des Satzes 2 erfüllt.

Satz 3. *Gilt für das Verfahren $A = (a_{k,n})$:*

(α) *$a_{0,n} \geqslant 0$ für $n \geqslant N_0$, $a_{0,n} > 0$ für unendlich viele n — und*

(β) *für jedes k existiert der endliche* $\lim\limits_{m\to\infty}\dfrac{\sum\limits_{n=m}^{\infty} a_{k,n}}{\sum\limits_{n=m}^{\infty} a_{0,n}} = c_k,$

so besitzt A eine Lückenfunktion.

Bemerkung 5. Ist $a_{0,n} > 0$ für $n \geqslant N_0$ und konvergiert die Folge $\left\{\frac{a_{k,n}}{a_{0,n}}\right\}$ für jedes k, so genügt $A = (a_{k,n})$ den Voraussetzungen des Satzes 3. Insbesondere sind diese erfüllt, wenn $a_{0,n} > 0$, $a_{k,n} \neq 0$ für $n \geqslant N_k$ und jede der Folgen $\left\{\frac{a_{k+1,n}}{a_{k,n}}\right\}$ $(k = 0, 1, \ldots)$ konvergent ist.

Bemerkung 6. Es sei $N_{p,k} = [1+2+\ldots+(p+1)] - (k+1)$. Das Verfahren $A = (a_{k,n})$, wo

$$a_{k,n} = \begin{cases} \dfrac{1}{2^p} & \text{für } k = 0;\ n = N_{p,0};\ p = 0, 1, \ldots, \\ 1 & \text{für } k = 1, 2, \ldots;\ n = N_{k,k}, \\ \dfrac{1}{3^p} & \text{für } k = 1, 2, \ldots;\ n = N_{p,k};\ p = k+1, k+2, \ldots, \\ 0 & \text{in übrigen Fällen,} \end{cases}$$

erfüllt die Bedingungen des Satzes 3, jedoch gibt es für A kein nichtschwächeres zeilenfinites permanentes Verfahren.

Es ist noch nicht entschieden, ob ein diese Eigenschaft besitzendes Verfahren, welches zugleich den Bedingungen der Bemerkung 5 genügt, existiert.

Satz 4. *Genügt das Verfahren $A = (a_{k,n})$ folgenden Voraussetzungen:*

(α) *$a_{0,n} \geqslant 0$ für $n \geqslant N_0$, $a_{0,n} > 0$ für unendlich viele n,*

(β) *für jedes k ist* $\overline{\lim\limits_{m\to\infty}}\left|\dfrac{\sum\limits_{n=m}^{\infty} a_{k,n}}{\sum\limits_{n=m}^{\infty} a_{0,n}}\right| < \infty,$

so kann man aus jeder wachsenden Folge der ganzen, nichtnegativen Zahlen eine Lückenfolge für das Verfahren A entnehmen.

Bemerkung 7. Gilt $a_{0,n} > 0$ für $n \geqslant N_0$ und ist jede der Folgen $\left\{\frac{a_{k,n}}{a_{0,n}}\right\}$ $(k = 0, 1, \ldots)$ beschränkt, so genügt $A = (a_{k,n})$ den Bedingungen des

Satzes 4. Insbesondere sind sie erfüllt, wenn $a_{0,n} > 0$, $a_{k,n} \neq 0$ für $n \geqslant N_k$ und jede der Folgen $\left\{\frac{a_{k+1,n}}{a_{k,n}}\right\}$ $(k = 0, 1, \ldots)$ beschränkt ist.

Bemerkung 8. Man kann nicht behaupten, dass. unter den Voraussetzungen des Satzes 4, das Verfahren A eine Lückenfunktion besitzt. Zum Beispiel, hat das Verfahren $A = (a_{k,n})$, wo

$$a_{k,n} = \begin{cases} 0 & \text{für } n < k, \\ C\dfrac{1}{2^n} & \text{für } n \geqslant k,\ n \neq p^2+k,\ (p = 0, 1, \ldots), \\ C\left(\dfrac{1}{2^n} + \dfrac{1}{2^{p^2}}\right) & \text{für } n = p^2+k,\ p = 0, 1, \ldots. \end{cases}$$

und $\frac{1}{C} = \sum_{p=0}^{\infty} \frac{1}{2^{p^2}}$, keine Lückenfunktion, obgleich es sogar den Bedingungen der Bemerkung 7 genügt.

SATZ 5. *Genügt das Verfahren $A = (a_{k,n})$ folgenden Voraussetzungen:*

(α) *für jedes k ändern die von Null verschiedenen $a_{k,n}$, von einer Stelle beginnend, das Vorzeichen nicht,*

(β) *für jedes k ist* $\lim\limits_{m\to\infty} \dfrac{\sum\limits_{n=m}^{\infty} a_{k+1,n}}{\sum\limits_{n=m}^{\infty} a_{k,n}} = \infty,$

so besitzt dieses Verfahren keine Lückenfolgen.

Bemerkung 9. Gilt (α) mit $a_{k,n} \neq 0$ für $n \geqslant N_k$, und $\lim\limits_{n\to\infty} \frac{a_{k+1,n}}{a_{k,n}} = \infty$ für jedes k, so erfüllt das Verfahren $A = (a_{k,n})$ die Bedingungen des Satzes 5.

Bemerkung 10. In den Sätzen 2, 3 und 4 genügt es, dass die Voraussetzung (β) mindestens für unendlich viele k erfüllt wird. Im Satz 5 muss (β) für alle $k = 0, 1, \ldots$ stattfinden; anderfalls könnte das Verfahren sogar den Bedingungen des Satzes 2 genügen, wie es einfache Beispiele zeigen.

Bemerkung 11. Sei $\{c_k\}$ eine gegebene Zahlenfolge, $0 \leqslant c_k < 1$ $(k = 0, 1, \ldots)$, $c_0 < c_1 < c_2 < \ldots$ und das Verfahren $A = (a_{k,n})$ folgendermassen erklärt:

$$a_{k,n} = 0 \quad \text{für} \quad n < 2k,$$

$$a_{k,2k} = a_{k,2k+1} = \frac{1}{2} \text{ für } \quad k = 0, 1, \ldots,$$

$$a_{k,2i} = \frac{1}{(2i)!} - \frac{1}{(2i+1)!}(2i+1)^{c_k} \qquad \text{für} \quad k = 0, 1, \ldots, i = k+1, k+2, \ldots,$$

$$a_{k,2i+1} = \frac{1}{(2i+1)!}(2i+1)^{c_k} - \frac{1}{(2i+2)!} \quad \text{für} \quad k = 0, 1, \ldots, \quad ; i = k+1, k+2, \ldots.$$

Das Verfahren A besitzt folgende Eigenschaft: Die Folge $\{n_\nu\}$ ist eine Lückenfolge für A genau dann, wenn fast alle n_ν gerade sind.

Hier ist $\lim\limits_{i\to\infty}\dfrac{\sum\limits_{n=2i}^{\infty} a_{k+1,n}}{\sum\limits_{n=2i}^{\infty} a_{k,n}} = 1$ und $\lim\limits_{i\to\infty}\dfrac{\sum\limits_{n=2i+1}^{\infty} a_{k+1,n}}{\sum\limits_{n=2i+1}^{\infty} a_{k,n}} = \infty$ für $k = 0, 1, \ldots$.

Ich möchte es nicht versäumen, an dieser Stelle Herrn Prof. S. Mazur meinen Dank für seine wertvolle Ratschläge auszusprechen.

MATEMATISCHES INSTITUT, POLNISCHE AKADEMIE DER WISSENSCHAFTEN
(INSTYTUT MATEMATYCZNY, PAN)

SCHRIFTTUM

[1] M. Eidelheit, *Zur Theorie der Systeme linearer Gleichungen I*, Studia Math. **6** (1936), 139—148; II, ibid. **7** (1938), 150—154.

[2] P. Erdös and G. Piranian, *Convergence fields of row-finite and row-infinite Toeplitz transformations*, Proc. Amer. Math. Soc. **1** (1950), 397—401.

[3] S. Mazur and W. Orlicz, *On linear methods of summability*, Studia Math. **14** (1954), 129—160.

[4] W. Meyer-König und K. Zeller, *Lückenumkehrsätze und Lückenperfektheit*, Math. Zeitschr. **66** (1956), 203—224.

[5] K. Zeller, *Theorie der Limitierungsverfahren*, Berlin—Göttingen—Heidelberg, 1958.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 5, 1961

MATHEMATICS

On the Freudenthal Compactification

by

R. ENGELKING

Presented by K. KURATOWSKI on March 28, 1961

H. Freudenthal proved [2], [3] that any metric, separable space X has a metric compactification $\widetilde{X}$ such that ind $(\widetilde{X} - X) \leqslant 0$*) if and only if X is a peripherically compact**) space. M. Smirnov showed [7] that this theorem was not true for arbitrary spaces. E. Sklarenko [6] has called X an $\mathfrak{S}$-space if each compact set $C \subset X$ possesses a countable basis of neighbourhoods in X and he has proved that the Freudenthal's theorem is still true for $\mathfrak{S}$-spaces. Moreover, in [6] the author constructed the compactification $\widetilde{X}$ such that ind $(\widetilde{X} - X) \leqslant 0$ for an arbitrary completely regular and peripherically compact X. In [1] other necessary and sufficient conditions for X to be a peripherically compact space are found. E. Sklarenko [6] has used the theory of proximity spaces, while the paper [1] bases upon the general theory of compactification of completely regular spaces constructed by the authors.

The purpose of this paper is to find a simple and direct construction of Freudenthal's compactification for peripherically compact $\mathfrak{S}$-spaces. That construction, analogous to the proof of Urysohn's metrization theorem, asserts the existence of a compactification $\widetilde{X}$ such that ind $(\widetilde{X} - X) \leqslant 0$ and weight $\widetilde{X} =$ weight X. For the sake of completness it will be proved (essentially as in [6]) that existence of the compactification $\widetilde{X}$, such that ind $(\widetilde{X} - X) \leqslant 0$, is sufficient for peripherically compactness of the space X.

LEMMA 1. *Let* $\{O_iC\}_{i=1}^{\infty}$ *denote a basis of neighbourhoods for* $C \subset X$ *and let* A *be an arbitrary set;*

$$\textit{if } A \cap O_iC \neq 0 \textit{ for } i = 1, 2, 3, \ldots \textit{ then } \overline{A} \cap C \neq 0.$$

Indeed, let us admit that $C \subset X - \overline{A}$. Then there exists j such that $C \subset O_jC \subset$ $\subset X - \overline{A}$ contrary to the supposition that $A \cap O_jC \neq 0$.

---

*) ind denotes the dimension in the sense of Menger-Urysohn.

**) A space is said to be *peripherically compact*, if it has the basis $\mathfrak{B} = \{U\}$ such that the boundary Fr (U) is compact for each $U \in \mathfrak{B}$.

Let X be an $\mathfrak{S}$-space with the basis $\mathfrak{B}$ consisting of sets with compact boundaries. It may be assumed that

$$\text{(1)} \qquad \text{If } V \in \mathfrak{B}, \text{ then } U \cap V \in \mathfrak{B},\ U \cup V \in \mathfrak{B} \text{ and } X - U \in \mathfrak{B},$$

$$\text{(2)} \qquad \overline{\overline{\mathfrak{B}}} = \text{weight } X.$$

The 2-uple (V, U), where $U, V \in \mathfrak{B}$, is said to be *a pair*, if

$$\text{(3)} \qquad V \subset \overline{V} \subset U,\ X - \overline{U} \neq 0.$$

A condensation of a pair (V, U) is a family $\{V_r\}$ consisting of elements from $\mathfrak{B}$ enumerated by the triadic-rational numbers $0 \leqslant r \leqslant 1$ and such that

$$\text{(4)} \qquad V = V_{1/3},\ U = V_{2/3},\ \overline{V}_r \subset V_{r'} \text{ for } r < r',$$

$$\text{(5)} \qquad (V_r - V_{r-1/3^i}) \cup (V_{r+1/3^i} - V_r) \subset O_i \operatorname{Fr}(V_r) \text{ for } i > n(r) + 1,$$

where $n(r)$ denotes the power of 3 in the denominator of the irreducible fraction r, and $\{O_i \operatorname{Fr}(V_r)\}_{i=1}^{\infty}$ — an arbitrary but fixed basis of neighbourhoods for the (compact) set $\operatorname{Fr}(V_r)$.

Lemma 2. *For each pair there exists a condensation.*

For the proof let us order the triadic-rational numbers in the infinite sequence: 0, 1, 1/3, 2/3, 1/9, 2/9, 4/9, ... Let (V, U) be an arbitrary pair and let $x_0 \in V$, $x_1 \in X - \overline{U}$; Let us take $V_0, V_1^* \in \mathfrak{B}$ such that

$$x_0 \in V_0 \subset \overline{V}_0 \subset V,\ x_1 \in V_1^* \subset \overline{V}_1^* \subset X - \overline{U}.$$

From (1) it follows that

$$\overline{U} \subset V_1 = X - \overline{V}_1^* \in \mathfrak{B}.$$

It is easy to observe that setting $V_{1/3} = V$, $V_{2/3} = U$ we have conditions (4) and (5) satisfied for $r = 0, 1, 1/3, 2/3$. Now, let us suppose that the sets V_r are defined for all r_j, where $j < k$, and that they are subject to the conditions (4) and (5). Let r_s and r_t denote these numbers r_j, where $j = 1, 2, \ldots, k-1$, for which the differences $r_k - r_j$ and $r_j - r_k$ admit the least possible positive value, respectively. It can be easily proved that if r_k can be expressed in the form of $r_j \pm 1/3^i$, where $i > n(r_j) + 1$ and $j < k$, then such a representation is unique. Moreover, $j = s$ or $j = t$ holds. We may assume without loss of generality that $r_k = r_s + 1/3^i$, where $i > n(r_s) + 1$. If $r_k = r_t - 1/3^i$, we proceed analogously. For each $p \in \operatorname{Fr}(V_s)$ let $V(p) \in \mathfrak{B}$ be a neighbourhood such that

$$\text{(6)} \qquad p \in V(p) \subset \overline{V(p)} \subset V_{r_t} \cap O_i \operatorname{Fr}(V_{r_s}).$$

Let $V(p_1), \ldots, V(p_l)$ compose a finite covering of the compact set $\operatorname{Fr}(V_s)$. Let us set

$$\text{(7)} \qquad V_{r_k} = V_{r_s} \cup \bigcup_{i=1}^{l} V(p_i).$$

Obviously, from (1) we have $V_{r_k} \in \mathfrak{B}$, but from (6) and (7) it follows that

$$\overline{V}_{r_k} \subset V_{r_t},\ V_{r_k}^{\cdot} - V_{r_s} \subset \mathrm{O}_i \operatorname{Fr}(V_{r_s})$$

which proves that conditions (4) and (5) remain satisfied.

Thus the proof of Lemma 2 is concluded.

The condensation $\{V_r\}$ of an arbitrary pair determines the function f given by the formula

$$f(x) = \begin{cases} \inf\limits_{x \in V_r} r & \text{for } x \in \bigcup\limits_{r \leqslant 1} V_r \\ 1 & \text{for } x \in X - V_1 \end{cases} \tag{8}$$

LEMMA 3. *Function f (determined by the condensation $\{V_r\}$) is continuous. Moreover, for every $A \subset X$ and triadic-rational number r we have*

$$\text{if } r \in \overline{f(A)}, \text{ then } \overline{A} \cap Fr(V_r) \neq 0. \tag{9}$$

The continuity of f is well known (see, e.g., [5], p. 114). Let us suppose that for some $j > n(r)+1$ the intersection $A \cap \mathrm{O}_j \operatorname{Fr}(V_r)$ is empty. In view of (5) it follows that

$$A \cap [(V_r \cap V'_{r-1/3^j}) \cup (V_{r+1/3^j} \cap V'_r) = 0.$$

By (4) it gives

$$A \subset (V'_r \cup V_{r-1/3^j}) \cap (V'_{r-1/3^j}) \cup V_r) = V'_{r+1/3^j} \cup V_{r-1/3^j}.$$

Comparing the last relation with (8) we get

$$\overline{f(A)} \subset \overline{f(V'_{r+1/3^j} \cup V_{r-1/3^j})} \subset \overline{f(V'_{r+1/3^j})} \cup \overline{f(V'_{r-1/3^j})} \subset [r+1/3^j, 1] \cup [0, r-1/3^j]$$

contrary to the supposition that $r \in \overline{f(A)}$. Therefore we have $A \cap \mathrm{O}_j \operatorname{Fr}(V_r) \neq 0$ for $j > n(r)+1$ and by Lemma 1 $\overline{A} \cap \operatorname{Fr}(V_r) \neq 0$.

FREUDENTHAL THEOREM. *The $\mathfrak{S}$-space X possesses a compactification $\widetilde{X}$ such that $\operatorname{ind}(\widetilde{X} - X) \leqslant 0$ and weight $\widetilde{X}$ = weight X if and only if X is peripherically compact.*

Proof of sufficiency. Let $\mathfrak{B}$ denote a basis of X consisting of sets with compact boundaries which satisfies (1) and (2). Let $\mathfrak{P}$ denote the family of all pairs (V, U) It is clear that $\overline{\overline{\mathfrak{P}}} = \mathfrak{p} =$ weight X. For each pair $p = (V, U)$ let V_r^p denote an arbitrary condensation of p and let f_p be the function determined by the condensation.

The function $F(x) = \{f_p(x)\}_{p \in \mathfrak{P}}$ maps X homeomorphically into the Tichonov cube $I^{\mathfrak{p}}$ (see [5], p. 116). Let us put $\widetilde{X} = \overline{F(X)} \subset I^{\mathfrak{p}}$. We shall prove that $\widetilde{X}$ is the required compactification. Since the weight of $\widetilde{X}$ is obviously equal to the weight of X, it remains to prove that $\operatorname{ind}(\widetilde{X} - X) \leqslant 0$. For this purpose it is sufficient to find a subbasis in $I^{\mathfrak{p}}$ consisting of open sets with boundaries non intersecting $\widetilde{X} - F(X)$. We shall prove that the subbasis consisting of the sets $U(p_0, r_1, r_2) =$

$=\{\{x_p\}: r_1 < x_{p_0} < r_2\}$, where $p_0 \in \mathfrak{P}$ and r_1, r_2 are triadic-rational numbers, has the above mentioned property. We have to verify that

(10) $$\textit{if } \mathfrak{x} = \{x_p\} \in \widetilde{X} \textit{ and } x_{p_0} = r, \textit{ then } \mathfrak{x} \in F(\mathrm{Fr}(V_r)).$$

Since $F(\mathrm{Fr}(V_r))$ is compact it suffices to show that for an arbitrary neighbourhood O of $\mathfrak{x}$ we have $O \cap F(\mathrm{Fr}(V_r)) \neq 0$. Let G be a neighbourhood of $\mathfrak{x}$ such that $\overline{G} \subset O$ and let us set

(11) $$G_j = G \cap \{\{x_p\}: |x_{p_0} - r| < 1/3^j\}.$$

Since G_j is the neighbourhood of $\mathfrak{x} \in \widetilde{X}$ there exists $x_j \in X$ such that $F(x_j) \in G_j$. Let $A = \bigcup_{j=1}^{\infty} \{x_j\}$. Lemma 3 (for the set A, number r and function f_{p_0}) implies $A \cap \cap \mathrm{Fr}(V_r) \neq 0$, hence

$$0 \neq F(\bar{A}) \cap F(\mathrm{Fr}(V_r)) \subset \overline{F(A)} \cap F(\mathrm{Fr}(V_r)) \subset O \cap F(\mathrm{Fr}(V_r)),$$

which completes the proof of sufficiency.

Proof of necessity. Let $\widetilde{X}$ denote the compactification of the $\mathfrak{S}$-space X. First we observe that each compact set $C \subset X$ is a G_δ-set in $\widetilde{X}$. Indeed, let O_iC denote a countable basis of neighbourhoods of C in X. We have $O_iC = H_i \cap X$ where H_i is open in $\widetilde{X}$. Then $\bigcap_{i=1}^{\infty} H_i = C$ holds. Otherwise, there exist $x \in \bigcap_{i=1}^{\infty} H_i - C$ and an open $V \subset \widetilde{X}$ such that $x \in V$, $\overline{V} \cap C = 0$. Consequently, because X is dense in $\widetilde{X}$, we have

$$(X \cap V) \cap O_iC = V \cap H_i \cap X \neq 0 \text{ for every } i$$

and, by Lemma 1, $X \cap (\overline{X \cap V}) \cap C \neq 0$, contrary to $\overline{V} \cap C = 0$.

Now we shall prove that the space $N = \widetilde{X} - X$ possesses the Lindelöf property. Let $\{V_\alpha\}_{\alpha \in A}$ denote the covering of N. It is clear that $V_\alpha = U_\alpha \cap N$ where U_α are open in $\widetilde{X}$. Let $C = \widetilde{X} - \bigcup_{\alpha \in A} U_\alpha$. Then we have $C \subset X$ and C is compact. Hence, by the above remarks, C is a G_δ-set. It follows that the set $U = \bigcup_{\alpha \in A} U_\alpha$ is an F_σ-set and it is a countable sum of compact sets. It means that there exists a countable family $\{U_{\alpha_i}\}$ such that $U = \bigcup_{i=1}^{\infty} U_{\alpha_i}$. Obviously, the family $\{V_{\alpha_i}\}_{i=1}^{\infty}$ forms a countable covering of the space N.

Let us suppose that $\widetilde{X}$ denotes the compactification of the $\mathfrak{S}$-space X such that $\mathrm{ind}\,(\widetilde{X} - X) \leqslant 0$. Let $x \in X$ and let U be the neighbourhood of x in $\widetilde{X}$. To prove the necessity of our Theorem it suffices to find a neighbourhood G of x in X such that

(12) $$x \in G \subset U,\ \mathrm{Fr}(G) = \overline{G} - G \subset X.$$

The space $N_1 = N \cup \{x\}$ possesses the Lindelöf property. Hence it is subject to the Sum Theorem for Dimension (for proof see, e.g., [4] p. 18). Therefore we

have ind $N_1 \leqslant 0$ and there exists such a closed-open neighbourhood H of x in N_1 that $H = H_1 \cap N_1$, $\bar{H}_1 \subset U$, where H_1 is open in $\tilde{X}$ and $N_1 \cap \bar{H} - H \quad 0$. Putting $C = \bar{H} \cap \overline{N_1 - H}$ we have

$$C \cap N_1 = \bar{H} \cap N_1 \cap \overline{N_1 - H} = H \cap (N_1 - H) = 0.$$

C is a compact subset of X, hence by the above remarks C is a G_δ—set in $\tilde{X}$. The complementary set $S = \tilde{X} - C$ is an F_σ-set in the compact space $\tilde{X}$, therefore S possesses Lindelöf property and it is normal. Since $H \cap S$ and $\overline{N_1 - H} \cap S$ are closed, disjoint subsets of S there exists a set G open in S, hence in $\tilde{X}$, such that

$$x \in \bar{H} \cap S \subset G \subset U, \quad S \cap \bar{G} \cap \overline{N_1 - H} = 0. \tag{13}$$

From (13) it follows that

$$N_1 - G \subset N_1 - H \cap S \subset N_1 \cap S - \bar{H} = N_1 - H,$$

$$\bar{G} \cap S \subset X - \overline{N_1 - H} \subset \tilde{X} - (N_1 - H)$$

and

$$N \cap \mathrm{Fr}(G) \subset (\bar{G} - G) \cap N_1 \subset (N_1 - G) \cap \bar{G} \cap S = 0$$

which completes the proof of (12) and of Freudenthal theorem.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY, PAN)

REFERENCES

[1] P. S. Aleksandrov and W. Ponomarev, Vestnik Mosk. Univ. **5** (1959), 93.
[2] H. Freudenthal, Annals of Math. **43** (1942), 261.
[3] — , Indag. Math. **13** (1951), 184.
[4] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton Univ. Press, 1948.
[5] J. L. Kelley, *General Topology*, Van Nostrand, 1955.
[6] E. Sklarenko, Doklady A.N., **120** (1958), 1200.
[7] Yu. M. Smirnov, Doklady A.N., **120** (1958), 1204.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 5, 1961

MATHEMATICS

Summability with Korovkin's Factors

by

R. TABERSKI

Presented by W. ORLICZ on April 4, 1961

1. Notation and auxiliary results. Given any sequence $\{u_k\}$ of real numbers, we write $(C, 1) \sum_{k=1}^{\infty} u_k = s$, when the series $\sum_{k=1}^{\infty} u_k$ is $(C, 1)$-summable to s. Similar notation will be used for series $\sum_{k=1}^{\infty} u_k(x)$ of functions.

In the theory of summability the following theorem of Hurwitz is known ([3], p. 45—6).

1.1. *A necessary and sufficient condition for*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \alpha_k^{(n)} u_k = s, \qquad (s \textit{ finite})$$

whenever

$$(C, 1) \sum_{k=1}^{\infty} u_k = s, \tag{1}$$

is that

$$\sum_{k=1}^{\infty} k\,|\alpha_k^{(n)} - 2\alpha_{k+1}^{(n)} + \alpha_{k+2}^{(n)}| \leqslant M \qquad (n = 1, 2, \ldots), \tag{2}$$

$$|k\alpha_k^{(n)}| \leqslant M^{(n)} \qquad (k, n = 1, 2, \ldots), \tag{3}$$

$$\lim_{n\to\infty} \alpha_k^{(n)} = 1 \qquad (k = 1, 2, \ldots), \tag{4}$$

where M and $M^{(n)}$ are positive constants, $M^{(n)}$ depends on n.

1.2. *It may be easily shown that the conditions* (2), (3) *and* (4) *remain also necessary and sufficient for*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \alpha_k^{(n)} u_k(x) = s(x) \qquad \textit{uniformly in a set } E,$$

whenever

$$(C, 1) \sum_{k=1}^{\infty} u_k(x) = s(x) \qquad \textit{uniformly in } E$$

and $u_k(x)$ $(k = 1, 2, \ldots)$ are bounded in E.

Denote by $L_{2\pi}$ the class of all 2π-periodic functions Lebesgue-integrable in the in terval $\langle -\pi, \pi\rangle$. Let

$$(5) \qquad S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{\infty} A_k(x)$$

be the Fourier series of $f \in L_{2\pi}$, and

$$(6) \qquad \widetilde{S}[f] = \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx) = \sum_{k=1}^{\infty} B_k(x)$$

its conjugate, and let

$$(7) \qquad \widetilde{f}(x; h) = -\frac{1}{2\pi}\int_h^{\pi} [f(x+t) - f(x-t)] \cot \tfrac{1}{2} t dt, \quad \widetilde{f}(x) = \lim_{h\to 0+} \widetilde{f}(x; h).$$

Now a theorem concerning the $(C, 1)$-summability of $\widetilde{S}[f]$ will be given.

1.3. *If a function $f \in L_{2\pi}$ is continuous in a closed interval $\langle a, b\rangle$, being continuous to the left at a and to the right at b, and if*

$$\lim_{h\to 0+} \widetilde{f}(x; h) = \widetilde{f}(x) \qquad \textit{uniformly in } \langle a, b\rangle,$$

then

$$(C, 1) \sum_{k=1}^{\infty} B_k(x) = \widetilde{f}(x) \qquad \textit{uniformly in } \langle a, b\rangle.$$

The proof of **1**.3. is similar to that of (3.20) and (2.21) in [5], pp. 92, 86—7.

Theorems formulated above and others are used in the next sections in connexion with investigation of a new method of summability introduced for Fourier series by Korovkin. The definition of this method will now be given.

The series $\sum_{k=1}^{\infty} u_k$ is called (K)-summable to s, in symbols

$$(K) \sum_{k=1}^{\infty} u_k = s,$$

if and only if

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \varrho_k^{(n)} u_k = s \qquad (s \textit{ finite}),$$

where

$$(8) \qquad \varrho_k^{(n)} = \frac{1}{2(n+2)}\left[(n-k+3)\sin\frac{(k+1)\pi}{n+2} - \right.$$
$$\left. -(n-k+1)\sin\frac{(k-1)\pi}{n+2}\right]\operatorname{cosec}\frac{\pi}{n+2} \qquad (k = 1, 2, \dots, n)$$

(see [1], pp. 75—8, [2], [4]). We extend this definition to functional series $\sum_{k=1}^{\infty} u_k(x)$.

2. Relation between methods $(C, 1)$ **and** (K). Simple calculation shows that Korovkin's factors $\varrho_k^{(n)}$ defined by (8) for $k \leqslant n$ and $\varrho_k^{(n)} = 0$ for $k \geqslant n+1$ are non-increasing with respect to k, more exactly

$$0 \leqslant \varrho_{k+1}^{(n)} < \varrho_k^{(n)} < 1 \qquad \text{if } 1 \leqslant k \leqslant n,$$

and satisfy the conditions (3) and (4) formulated for $\alpha_k^{(n)}$.

It is easy to check that

$$\varrho_k^{(n)} - 2\varrho_{k+1}^{(n)} + \varrho_{k+2}^{(n)} = \frac{2}{n+2}\left[\sin\frac{(k+1)\pi}{n+2} - \right.$$
$$\left. -(n-k+1)\sin\frac{\pi}{n+2}\cos\frac{(k+1)\pi}{n+2}\right]\tan\frac{\pi}{2(n+2)}$$

for $1 \leqslant k \leqslant n-1$. Hence

$$\sum_{k=1}^{\infty} k\,|\varrho_k^{(n)} - 2\varrho_{k+1}^{(n)} + \varrho_{k+2}^{(n)}| = \sum_{k=1}^{n-1} k\,|\varrho_k^{(n)} - 2\varrho_{k+1}^{(n)} + \varrho_{k+2}^{(n)}| + n\varrho_n^{(n)} \leqslant$$
$$\leqslant \frac{2}{n+2}\left[\sum_{k=1}^{n-1} k + n\sin\frac{\pi}{n+2}\sum_{k=1}^{n-1} k\right]\tan\frac{\pi}{2(n+2)} + \frac{3\pi n}{2(n+2)^2}\operatorname{cosec}\frac{\pi}{n+2} < 3\pi.$$

Therefore, $\varrho_k^{(n)}$ satisfies also the condition (2) and, by 1.1, we obtain the following result.

2.1. *For any sequence* $\{u_k\}$ *the relation* (1) *implies*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\varrho_k^{(n)} u_k = s.$$

Remark. In the last statement the relation (1) can be replaced by

$$(C, 3)\sum_{k=1}^{\infty} u_k = s.$$

3. (K)**-summability of series** $S[f]$ **and** $\widetilde{S}[f]$. From 2.1, 1.2 and (3.9), (3.4) of [5] (pp. 90, 89) the next theorem follows.

3.1. *Let* $f \in L_{2\pi}$. *Then the series* (5) *is* (K)*-summable to* $f(x)$, *i.e.*

$$\lim_{n\to\infty}\left(\frac{a_0}{2} + \sum_{k=1}^{n}\varrho_k^{(n)} A_k(x)\right) = f(x), \tag{9}$$

for every x *at which*

$$\lim_{h\to 0+}\frac{1}{h}\int_0^h |f(x+t) - 2f(x) + f(x-t)|\,dt = 0.$$

If f *is continuous in a closed interval* $\langle a, b\rangle$, *being also continuous to the left at* a *and to the right at* b, *the relation* (9) *holds uniformly in* $\langle a, b\rangle$ (compare [1], pp. 71, 75—8, 127—9, 131, 175).

By **2**.1, **1**.2, **1**.3 and (3.23) of [5] (p. 92) we have

3.2. *Let $f \in L_{2\pi}$. Then the series* (6) *is* (*K*)-*summable to* $\tilde{f}(x)$ (*see* (7)), *i.e.*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \varrho_k^{(n)} B_k(x) = \tilde{f}(x), \tag{10}$$

for almost all x.

If all assumptions of **1**.3 *are satisfied, the relation* (10) *holds uniformly in* $\langle a, b\rangle$.

DEPARTMENT OF MATHEMATICS, A. MICKIEWICZ UNIVERSITY, POZNAŃ
(KATEDRA MATEMATYKI, UNIWERSYTET im. A. MICKIEWICZA, POZNAŃ)

REFERENCES

[1] P. P. Korovkin, *Linear operators and the approximation theory* [in Russian], Fizmatgiz, Moscow, 1959.

[2] — , *On some asymptotic property of positive methods of summation of the Fourier series* [...] [in Russian], Usp. mat. nauk, **13** (1958), 99—103.

[3] Ch. N. Moore, *Summable series and convergence factors*, AMS. Coll. Publ., XXII, New York, 1938.

[4] I. M. Petrov, *The order of approximation of functions of the class Z by means of some polynomial operators* [in Russian] Usp. mat. nauk, **13** (1958), 127—131.

[5] A. Zygmund, *Trigonometric series*, I Cambridge, 1959.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 5, 1961

THEORETICAL PHYSICS

On the Damping Problem in Quantum Theory

by

B. MIELNIK

Presented by W. RUBINOWICZ on March 4, 1961

X — is the Hilbert space, $\psi, \varphi, \ldots$ are vectors in X, (ψ, φ) — is a scalar product. $|\psi| = (\psi, \psi)^{1/2}$ — is the modulus of the vector. $A, B, \ldots$ are linear operators in X. $D(A)$ — is a domain of the A-operator.

$$(\varphi_n \to \varphi) \stackrel{df}{\equiv} (\lim_{n\to+\infty} |\varphi_n - \varphi| = 0),$$

$$(A_n \to A) \stackrel{df}{\equiv} (\lim_{n\to\infty} |A_n \varphi - A\varphi| = 0 \text{ for } \varphi \in D(A)).$$

$\mathfrak{M}_{||}, \mathfrak{M}_{\perp}$ are the subspaces X, $\mathfrak{M}_{||} \perp \mathfrak{M}_{\perp}$, $\mathfrak{M}_{||} \oplus \mathfrak{M}_{\perp} = X$;

$P_{||}, P_{\perp}$ are the operators of the normal (perpendicular) projections on $\mathfrak{M}_{||}$ and $\mathfrak{M}_{\perp}$, respectively;

H is the selfadjoined operator in X;

$P_{||}(t) = e^{iHt} P_{||} e^{-iHt}$ is the projection operator on the subspace $e^{iHt} \cdot \mathfrak{M}_{||}$.

1. Formulation

Let us have $\psi_0 \in \mathfrak{M}_{||}$. For the vector function $\psi_{||}(t) = P_{||} e^{-iHt} \psi_0$ the damping problem is said to have a solution, if there is a complex number λ and a non-zero vector $\varphi_0 \in \mathfrak{M}_{||}$, such that $e^{\lambda t} \psi_{||}(t) \to \varphi_0$ for $t \to +\infty$.

It may happen that there is a base $\{\psi^0, \psi^1, \ldots\}$ of the vectors of the space $\mathfrak{M}_{||}$ such that for each of the functions $\psi_{||}^n(t) = P_{||} e^{-iHt} \psi^n$ the damping problem has a solution and these solutions are represented by λ_n numbers and $\varphi^n \in \mathfrak{M}_{||}$ vectors respectively. Let in this case Λ and C be operators defined by the following relations:

$$\Lambda \psi^n = \lambda_n \psi^n, \quad C\psi^n = \varphi^n, \quad n = 0, 1, \ldots .$$

Then $P_{||} e^{-iHt} e^{\Lambda t} P_{||} \psi^n \xrightarrow[t\to+\infty]{} C\psi^n \; n = 0, 1, \ldots$

In view of this fact the damping problem may in general be formulated as follows:

The damping problem is said to have a solution for the subspace $\mathfrak{M}_{||}$, if there are Λ and C operators, defined in $\mathfrak{M}_{||}$, C — reversible, C, C^{-1} limited, such that:

(1) $$P_{||} e^{-iHt} e^{\Lambda t} P_{||} \to C \quad \textit{for} \quad t \to +\infty.$$

2. Uniqueness

Λ and C operators are not determined uniquely by the above condition. The following relation, however, may be noted:

If Λ, C and Λ', C' are two solutions of the damping problem, then

$$C\,\Lambda\,C^{-1} = C'\,\Lambda'\,C'^{-1}. \tag{2}$$

Proof. From the relation:

$$P_{||}\,e^{-iHt}\,e^{\Lambda' t}\,P_{||}\cdot P_{||}\,e^{-\Lambda' t}\,e^{\Lambda t}\,P_{||} = P_{||}\,e^{-iHt}\,e^{\Lambda t}\,P_{||} \underset{t\to+\infty}{\longrightarrow} C$$

it follows that

$$e^{-\Lambda' t}\,e^{\Lambda t} \underset{t\to+\infty}{\longrightarrow} C'^{-1}\,C.$$

Now let $\varphi \in D(\Lambda)$, $\psi \in D(\Lambda'^{*})$. Let us form a function $\chi(t) = (\psi, e^{-\Lambda' t}\,e^{\Lambda t}\,\varphi)$.

Therefore $\chi(t) \underset{t\to+\infty}{\longrightarrow} (\psi, C'^{-1}\,C\,\varphi)$

and

$$\chi'(t) = (-\Lambda'^{*}\,\psi, e^{-\Lambda' t}\,e^{\Lambda t}\,\varphi) + (\psi, e^{-\Lambda' t}\,e^{\Lambda t}\,\Lambda\varphi) \to$$
$$\underset{t\to+\infty}{\longrightarrow} (-\Lambda'^{*}\,\psi, C'^{-1}\,C\varphi) + (\psi, C'^{-1}\,C\Lambda\varphi).$$

Thus

$$-(\Lambda'^{*}\psi, C'^{-1}\,C\varphi) + (\psi, C'^{-1}\,C\Lambda\varphi) = \lim_{T\to+\infty}\int_{T}^{T+1}\chi'(t)\,dt =$$
$$= \lim_{T\to+\infty}\{\chi(T+1) - \chi(T)\} = 0.$$

It follows that $C'^{-1}\,C\varphi \in \Lambda(\Lambda'^{**})$ and $\Lambda'^{**}\,C'^{-1}\,C\varphi = C'^{-1}\,C\Lambda\varphi$. As this relation holds for each $\varphi \in D(\Lambda) = D(C'^{-1}\,C\Lambda)$, then $C'^{-1}\,C\Lambda \subseteq \Lambda'^{**}\,C'^{-1}\,C$ or $C\Lambda C^{-1} \subseteq C'\,\Lambda'^{**}\,C'^{-1} = (C'\,\Lambda'\,C'^{-1})^{**}$.

Thus $C\Lambda C^{-1}$ and $C'\Lambda' C'^{-1}$ operators differ one from another unessentially: they have the common extension.

On solving the concrete damping problem we look for the eigenvalues of the Λ operator, which are connected with the rates of disappearance of the states. As the spectra of the Λ and $C\,\Lambda\,C^{-1}$ operators are identical, the former as well as the latter is equally well-suited to the investigation of the "life times". $C\,\Lambda\,C^{-1}$ operator has, moreover, the advantage of being uniquely defined.

3. The equation for ψ

It will be assumed that

$$\|P_{\perp}\,P_{||}\,(-t)\,P_{\perp}\| < 1 \tag{3}$$

for large t. We have

$$\psi_{\perp} = P_{\perp}\,e^{-iHt}\,P_{||}\,\psi_0 = P_{\perp}\,e^{-iHt}\,P_{||}\,e^{iHt}\,(\psi_{||} + \psi_{\perp}) =$$
$$= P_{\perp}\,P_{||}\,(-t)\,P_{\perp}\,\psi_{\perp} + P_{\perp}\,P_{||}\,(-t)\,P_{||}\,\psi_{||}.$$

In view of (3) $1 - P_\perp P_{||}(-t) P_\perp$ operator is reversible, and thus

$$\psi_\perp = B(t)\, \psi_{||}, \tag{4}$$
$$\text{where } B(t) = [1 - P_\perp P_{||}(-t) P_\perp]^{-1} P_\perp P_{||}(-t).$$

The above equation makes it possible to find out the differential equation satisfied by $\psi_{||}$, namely

$$\frac{d\psi_{||}}{dt} = \frac{d}{dt}\{P_{||} e^{-iHt} \psi_0\} = -iP_{||} H e^{-iHt} \psi_0 =$$
$$= -i\{P_{||} HP_{||} \psi_{||} + P_{||} HP_\perp \psi_\perp\} = A(t)\, \psi_{||}$$

where

$$A(t) = -i\{P_{||} HP_{||} + P_{||} HP_\perp B(t)\}. \tag{5}$$

4. The relation between the $C\varLambda C^{-1}$ operator and the problem of the asymptotic equation for $\psi_{||}$

It will be assumed that $\varLambda$, C solve the damping problem on $\mathfrak{M}_{||}$. The following theorem holds:

If there is an A operator determined in $\mathfrak{M}_{||}$ and such that $A^*(t) \underset{t\to+\infty}{\to} A^*$, then

$$-C\varLambda C^{-1} = A. \tag{6}$$

The proof goes along the same lines as for (2).

This partially accounts for the fact why $C\varLambda C^{-1}$ operator is well-defined. At the same time it may be assumed that by resigning of a number of particularly odd cases, the damping theory might be constructed with the aid of the $A = (\lim\limits_{t\to+\infty} A^*(t))^*$ operator.

There is, however, a strong presumption that if the damping problem has a solution (in the sense of (1)), then the limit of the $A^*(t)$ operator may exist only in the trivial case, when $\varLambda$ spectrum lies completely on the imaginary axis.

5. The connection between the $C\varLambda C^{-1}$ operator and the K—R*) theory of the damping

The way of solving the damping problem developed by Królikowski and Rzewuski is an approximate method of calculating $C\varLambda C^{-1}$ operator. The exactness of the above method is first of all bound with the formulation of the problem represented by the relation (1).

Let $\varLambda$, C satisfy (1). We shall denote $U_{||}(t) = P_{||} e^{-iHt} P_{||}$. It has been proved in K—R's works that $U_{||}$ satisfies the equation

$$\left\{\frac{d}{dt} + iP_{||} P_{||}\right\} U_{||}(t) = \int_0^t K(\tau)\, U_{||}(t-\tau)\, d\tau, \tag{7}$$

where

$$K(\tau) = P_{||} H e^{-iP_\perp HP_\perp \tau} HP_{||}.$$

*) Królikowski—Rzewuski [1].

By multiplying the right-hand side of this equation by $e^{At} C^{-1}$, we get

$$(8) \qquad \frac{d}{dt}\{U_{\|}(t)\, e^{At}\}\, C^{-1} + iP_{\|}\, H P_{\|} \cdot U_{\|}(t)\, e^{At}\, C^{-1} + - U_{\|}(t)^{At}\, C^{-1}\, CAC^{-1} = \\ = \int\limits_0^t K(\tau)\, U_{\|}(t+\tau)\, e^{A(t-\tau)}\, C^{-1}\, C\, e^{A\tau}\, C^{-1}\, d\tau .$$

The expressions $U_{\|}(t)\, e^{At}$, $U_{\|}(t-\tau)\, e^{A(t-\tau)} \to C$ for $t \to +\infty$. Hence, for large t we have in approximation

$$(9) \qquad \frac{d}{dt}\{U_{\|}(t)\, e^{At}\}\, C^{-1} + iP_{\|}\, H P_{\|} = CAC^{-1} \sim \int\limits_0^t K(\tau)\, e^{CAC^{-1}\tau}\, d\tau .$$

Let us try to pass to the limit with $t \to +\infty$ in the integral appearing on the right-side of (9). If this limit does not exist in a normal way, then by

$$\int\limits_0^{+\infty} K(\tau)\, e^{CAC^{-1}\tau}\, d\tau = \lim_{t\to+\infty} \int\limits_0^t K(\tau)\, e^{CAC^{-1}\tau}\, d\tau$$

we shall understand the mean of the oscillations of the integral $\int\limits_0^t$ when $t \to +\infty$ according to the equation

$$(10) \qquad \lim_{t\to+\infty} \varphi(t) \overset{df}{=} \lim_{t,T\to+\infty} \frac{1}{T} \int\limits_t^{t+T} \varphi(\tau)\, d\tau .$$

Since $U_{\|}(t)\, e^{At} \to C = \text{const}$, the operator function $d/dt\, \{U_{\|}(t)\, e^{At}\}$ "oscillates round zero point" so that the mean of the oscillations in the sense of (10) equals zero.

In view of (9) the above defined integral is given by

$$(11) \qquad \int\limits_0^{+\infty} K(\tau)\, e^{CAC^{-1}\tau}\, d\tau = - CAC^{-1} + iP_{\|}\, H P_{\|} .$$

By denoting $- C A C^{-1} = - i\{P_{\|}\, H P_{\|} + V\}$ we get the basic equation in the K—R theory, namely

$$V = -i \int\limits_0^{+\infty} K(\tau)\, e^{i\{P_{\|} H P_{\|} + V\}\tau}\, d\tau .$$

The type of the approximation in the passage from Eq. (8) to (9) requires special treatment. The difference between the right-hand sides of these equations is given by

$$\int\limits_0^t K(\tau)\, \{U_{\|}(t-\tau)\, e^{A(t-\tau)} - C\}\, e^{A\tau}\, d\tau .$$

If $U_{\|}(t')\, e^{At'} \to C$ quickly enough for $t' = t - \tau \to +\infty$, then in this expression the integration in the range $(0, t)$ may be substituted for by that in a certain range $(t-\delta, t)$, where δ is not dependent on t.

Thus the integral

$$\int_{t-\delta}^{t} K(\tau)\,\{U_{||}(t-\tau)\,e^{\Lambda(t-\tau)}-C\}\,e^{\Lambda\tau}\,d\tau$$

presents the difference between the right-hand sides of (8) and (9). The function in the brackets $\{\quad\}$ does not run to $t \to +\infty$, $\tau \in (t-\delta, t)$ to zero, but takes on always the same finite values. It is therefore difficult to assume that the above integral runs to 0 for $t \to +\infty$. The approximation in the K—R theory consists essentially in the fact that for large t the integral within the small range $(t-\delta, t)$ is neglected as a small share to the integral appearing on the right-hand side of (8) within a large range $(0, t)$.

INSTITUTE OF THEORETICAL PHYSICS, UNIVERSITY, WARSAW
(INSTYTUT FIZYKI TEORETYCZNEJ, UNIWERSYTET, WARSZAWA)

REFERENCES

[1] W. Królikowski and J. Rzewuski, Nuovo Cimento, 3 (1956), 260; ibid. 4 (1956), 1212.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 5, 1961

THEORETICAL PHYSICS

On the Angular Momentum Weight Factor in the Statistical Theory of Multiple Particle Production. II

by

ZIRO KOBA

Presented by L. INFELD on March 14, 1961

1. Introduction

In the first part of this work [1] the angular momentum weight factor Z_n of the statistical theory of multiple particle production has been defined by

$$\mathcal{D}^{(l_1)} \times \mathcal{D}^{(l_2)} \times \dots \times \mathcal{D}^{(l_{n-1})} = \sum_{\lambda} Z_n(l_1, l_2, \dots, l_{n-1}, \lambda)\, \mathcal{D}^{(\lambda)}, \tag{1.1}$$

l's and λ being integers or half-odd integers, and it has been shown that Z_n can be obtained by the relations,

$$Z_n(l_1, l_2, \dots, l_{n-1}, l_n) = F^{(n-1)}_{l_1, \dots, l_{n-1}}(l_n) - F^{(n-1)}_{l_1, \dots, l_{n-1}}(l_n + 1), \tag{1.2}$$

$$= F^{(n)}_{l_1, \dots, l_n}(0) - F^{(n)}_{l_1, \dots, l_n}(1), \tag{1.3}$$

$$= F^{(n-2)}_{l_1, \dots, l_{n-2}}(l_n - l_{n-1}) - F^{(n-2)}_{l_1, \dots, l_{n-2}}(l_n + l_{n-1} + 1), \tag{1.4}$$

where $F^{(n)}$ is defined by

$$\chi^{(l_1)}(\varphi) \cdot \chi^{(l_2)}(\varphi) \dots \chi^{(l_n)}(\varphi) = \sum_{2\eta=0}^{\infty} F^{(n)}_{l_1, \dots, l_n}(\eta) \exp(i\,\eta\,\varphi), \tag{1.5}$$

with

$$\chi^{(l_i)}(\varphi) = \sum_{\mu=-l_i}^{l_i} \exp(i\,\mu\,\varphi) = \frac{\sin\,(l_i + \frac{1}{2})\,\varphi}{\sin \dfrac{\varphi}{2}}. \tag{1.6}$$

In this second part of the work an asymptotic expression of Z_n for $n \gg 1$*) and $\sum_{i=1}^{n} l_i^2 \gg l_j^2$ (for any j except two**)) will be derived based on the above relations

*) When some of l_i's are equal to zero, they do not contribute at all to the behaviour of Z_n. Therefore the number of these vanishing arguments must be subtracted from n in order to get the effective value of n, which should be much larger than one in order that our approximation is valid.

**) In other words, one or two of the arguments can be very large. See § 3.

and applying Edgeworth's expansion [2] to $F^{(n)}(y)$. Such an asymptotic form can be used for approximate evaluation of the multiplicity and the angular distribution of secondary particles created at superhigh energy collision.

2. Behaviour of F^n for a large n

The expression (1.6), when multiplied by $(2l_i+1)^{-1}$, can be regarded as the characteristic function*) of a random variable x_i, whose probability has an equal value $(2l_i+1)^{-1}$ at discrete points $x_i = -l_i, -l_i+1, \ldots, l_i-1, l_i$, and vanish otherwise. The left-hand side of (1.5) then represents, apart from a normalization factor $\prod_{i=1}^{n}(2l_i+1)^{-1}$, the characteristic function of the sum of these x_i's, and so the expression $\prod_{i=1}^{n}(2l_i+1)^{-1} F^{(n)}_{l_1,\ldots,l_n}(y)$ can be identified with the probabilities of the random variable $y = \Sigma x_i$. Thus we shall apply, in order to see its behaviour for a large value of n, the method of Edgeworth's series [2], which is known in the theory of probability**).

For this purpose we introduce the moments of the above-mentioned normalized distribution of x_i as follows.

$$\mu_k^{(i)} = \frac{1}{2l_i+1}\sum_{r=0}^{2l_i}(-l_i+r)^k, \quad (k=0,1,2,\ldots), \tag{2.1}$$

or, more explicitly,

(2.2)***)
$$\left|\begin{array}{ll} \mu_{2m+1}^{(i)} &= 0, \ (m=0,1,2,\ldots), \\ \mu_0^{(i)} &= 1, \\ \mu_2^{(i)} &= \frac{1}{3} l_i(l_i+1), \\ \mu_4^{(i)} &= \frac{1}{15} l_i(l_i+1)(3l_i^2+3l_i-1). \end{array}\right.$$

From these moments we obtain the semi-invariants of Thiele [2] for our distributions:

(2.3)
$$\left|\begin{array}{ll} \varkappa_{2m+1}^{(i)} &= 0, \ (m=0,1,2,\ldots), \\ \varkappa_2^{(i)} &= \mu_2^{(i)} = \frac{1}{3} l_i(l_i+1), \\ \varkappa_4^{(i)} &= \mu_4^{(i)} - 3-(\mu_2^{(i)})^2 \\ & \quad -\frac{1}{15} l_i(l_i+1)(2l_i^2+2l_i+1). \end{array}\right.$$

*) The terminology here follows that of [2].

**) In fact, we can imagine that the projection of each angular momentum vector on a certain axis is randomly distributed from $-l_i$ to l_i; and we are looking for the frequency for the sum of these independently distributed projections to take certain values.

***) Notice that these relations hold for half-odd integer values of l_i as well.

As is well known, $\varkappa_2$ represents the dispersion (or the standard deviation), while $\varkappa_4/(\varkappa_2)^2$ gives the so-called coefficient of excess. The negative value of the latter reflects the fact that the frequency "curve" of each x_i, which could be imagined to be rectangular, is flatter than the Gaussian curve.

With the help of the semi-invariants for individual x_i's we easily find the corresponding quantities for the standardized composite variable $\sum_{i=1}^{n} x_i/\sigma$,

$$(2.4)\qquad \begin{cases} \varkappa_{2m+1} = 0, \qquad (m = 0, 1, 2, \ldots) \\[2ex] \varkappa_2 = \dfrac{1}{3} \displaystyle\sum_{i=1}^{n} l_i(l_i+1) \equiv \sigma^2, \\[2ex] \varkappa_4 = -\dfrac{1}{15} \displaystyle\sum_{i=1}^{n} l_i(l_i+1)(2l_i^2+2l_i+1), \end{cases}$$

and thus write down the asymptotic expansion for $F^n(y)$ for a large value of n*) (for details and references see [2]):

$$(2.5)\qquad \frac{\sigma \cdot F^n(y)}{\prod\limits^{n}(2l_i+1)} \approx \varphi\left(\frac{y}{\sigma}\right) + \frac{1}{4!}\frac{\varkappa_4}{(\varkappa_2)^2}\varphi^{(4)}\left(\frac{y}{\sigma}\right) + O(n^{-2}),$$

where

$$(2.6)\qquad \begin{cases} \varphi(x) \equiv \dfrac{1}{\sqrt{2\pi}} \exp\left(-\dfrac{x^2}{2}\right), \\[2ex] \varphi^{(4)}(x) \equiv \dfrac{d^4}{dx^4}\varphi(x) = H_4(x) \cdot \varphi(x), \end{cases}$$

or, more explicitly,

$$(2.7)\qquad \begin{cases} F(y) \approx \sqrt{\dfrac{3}{2\pi}} \dfrac{\prod\limits^{n}(2l_i+1)}{\left\{\sum\limits^{n} l_i(l_i+1)\right\}^{1/2}} \exp\left(-\dfrac{3y^2}{2\sum\limits^{n} l_i(l_i+1)}\right) \times \\[3ex] \times \left[1 - \dfrac{3}{40}\dfrac{\sum\limits^{n} l_i(l_i+1)(2l_i^2+2l_i+1)}{\left\{\sum\limits^{n} l_i(l_i+1)\right\}^2}\left\{\dfrac{3y^4}{\left\{\sum\limits^{n} l_i(l_i+1)\right\}^2} - \dfrac{6y^2}{\sum\limits^{n} l_i(l_i+1)} + 1\right\}\right]. \end{cases}$$

3. Asymptotic expression of Z_n for large n

From (2.7) we obtain an approximate value of $Z_n(l_1, l_2, \ldots, l_n)$ either by (1.2) or (1.3) or (1.4). The choice of these alternatives depends on the problem we are interested in. If the symmetry property of Z's with respect to all the arguments is to be maintained in the approximate expression, we have to use (1.3). In many a practical application, however, our problem is to evaluate $Z_n(l_1, \ldots, l_{n-1}, l_n)$ for a given set of $l_1, l_2, \ldots, l_{n-1}$ and for all possible values of l_n; that is to say, l_n can be as large as $(l_1+l_2+\ldots+l_{n-1})$. Since the essential conditions for the expression to be a good description are*)

$$(3.1)\qquad n \gg 1,$$

*) We can go to further approximation, if we wish, and obtain the explicit form of the terms of order n^{-2}, but in most of the cases (2.5) appears practically sufficient.

where $l=0$ arguments should not be included in n, and

(3.2) $$\varkappa_2 \gg \varkappa_2^{(i)}\,,\ \text{for any } i\,,$$

it turns out more convenient in such a case to employ the relation (1.2) or (1.4). The latter can be applied even to a case where one of l_i's $(i=1, 2, ..., n-1)$ is very large compared to all others, or, in other words, when two of the arguments in $Z_n(l_1, l_2, ..., l_{n-1}, l_n)$ are very large compared to the others. (See the example (b) of Table I. Here the evaluation by (1.4)*) yields good result, while (1.2) gives very poor approximation).

Some numerical examples are shown in Table I. Although the effective numbers n in these simple cases are small, it is seen that for practical application our formula would give sufficient approximation. (The least favourable case is when we have three arguments of Z very large compared to all the others, as is shown by (c) of this Table. But in this example the error is still about 10—20%, which is not fatal for our purpose)

TABLE I

Numerical examples of the approximate evaluation of Z

(a) $Z_7(1, 1, 1, 1, 1, 2, L)$ evaluated by (1.2)

L	1st approx.	2nd approx.	Exact value
0	18.8	15.5	15
1	46.8	40.6	40
2	54.0	50.8	50
3	43.5	44.8	45
4	26.7	29.4	30
5	12.9	14.4	15
6	5.1	5.2	5
7	1.6	1.2	1
8	0.4	0.1	0

(b) $Z_7(0, 0, \frac{1}{2}, 1, 1, 10, L)$. (Effective $n=5$.)

L	Evaluated by (1.2).		Evaluated by (1.4).		Exact value
	1st approx.	2nd approx.	1st approx.	2nd approx.	
11/2	2.4	1.7	0.0	0.0	0
13/2	2.3	1.9	0.1	0.1	0
15/2	2.1	2.1	0.8	0.9	1
17/2	2.0	2.1	2.8	3.0	3
19/2	1.7	2.0	5.2	4.9	5
21/2	1.4	1.9	5.2	4.9	5
23/2	1.1	1.6	2.8	3.0	3
25/2	0.9	1.3	0.8	0.9	1
27/2	0.7	1.0	0.1	0.1	0
29/2	0.5	0.7	0.0	0.0	0

*) We denote then this largest one by l_{n-1} and apply (2.7) to $F^{(n-2)}_{l_1, ..., l_{n+2}}$ so that (3.2) can now be better satisfied.

(c) $Z_7(1, 1, 1, 1, 10, 10, 10)$. Exact value: 81

Evaluated by (1.2).		Evaluated by (1.3).		Evaluated by (1.4).	
1st approx.	2nd approx.	1st approx.	2nd approx.	1st approx.	2nd approx.
107.7	101.1	123.0	93.7	106.2	93.3

For a very crude estimation we may take the first term only in (2.7) and further replace the difference $F(l_n) - F(l_n+1)$ in (1.2) by a negative differential $-\partial F/\partial y$ at $y = l_n + \frac{1}{2}$, and thus get

$$Z_n(l_1, \ldots, l_{n-1}, l_n) \sim \frac{3\sqrt{3}}{\sqrt{2\pi}} \frac{\prod^{n-1} (2l_i+1)}{\left\{\sum^{n-1} l_i(l_i+1)\right\}^{3/2}} (l_n + \tfrac{1}{2}) \exp\left(-\frac{3(l_n+\frac{1}{2})^2}{2\sum^{n-1} l_i(l_i+1)}\right). \tag{3.3}$$

If, in particular, we put

$$l_1 = l_2 = \ldots = l_{n-1} \equiv l, \quad l_n \equiv L, \tag{3.4}$$

then (3.3) is reduced to

$$Z_n(l, \ldots, l, L) \approx \frac{3\sqrt{3}(2l+1)^{n-1}(L+\frac{1}{2})}{\sqrt{2\pi}(n-1)^{3/2} l^{3/2}(l+1)^{3/2}} \exp\left(-\frac{3(L+\frac{1}{2})^2}{2(n-1)l(l+1)}\right). \tag{3.5}$$

Regarded as a function of L, this expression reaches its maximum when

$$L = \sqrt{\frac{(n-1)l(l+1)}{3}} - \frac{1}{2}, \tag{3.6}$$

or for $n \gg 1$ and $l \gg 1$ (high energy peripheral collision),

$$L \sim \sqrt{\frac{n}{3}} \cdot l. \tag{3.7}$$

This is the most probable value of the resultant angular momentum when we have to combine at random a large number of identical (fairly large) angular momenta l. A similar result was once obtained by Ezawa [3] with the help of a more intuitive reasoning based on an analogy to the problem of random walk.

I express my gratitude to Professors L. Infeld and M. Danysz and to Dr J. Dąbrowski for their hospitality extended to me. I am also grateful to Mr. M. Majewski of the Łódź University for his comments.

This paper was written while on leave from Research Institute for Fundamental Physics, Kyoto University.

APPENDIX

Method of direct calculation of Z_n for smaller n.

A practical method of exact, direct calculation of Z_n for smaller values of n is shown below by examples.

To evaluate Z_5 (1, 3, 5, 6, L) for all values of L

1) take the arguments (1.3. 5), leaving L and the largest argument, 6.

2) Construct the series of numbers corresponding to $F^{(3)}_{1,2,5,}(x)$ step by step in the following way:

$$F^{(3)}_{1,3,5,}(x) \sum_{y} \sum_{z=-1}^{1} F_5^{(1)}(x-y)\, F_3^{(1)}(y-z)\, F_1^{(1)}(z) =$$

$$= \sum_{y} [F_5^{(1)}(x-y)\, F_3^{(1)}(y+1) + F_3^{(1)}(x-y)\, F_3^{(1)}(y) + F_5^{(1)}(x-y)\, F_3^{(1)}(y-1)] =$$

$$= \sum_{y-1=-3}^{y-1=3} F_5^{(1)}(x-y) + \sum_{y=-3}^{y=3} F_5^{(1)}(x-y) + \sum_{y+1=-3}^{y+1=3} F_5^{(1)}(x-y).$$

Thus for $x = -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, we get the series:

1, 3, 6, 9, 12, 15, 18, 20, 21, 21, 21, 20, 18, 15, 12, 9, 6, 3, 1.

3) Make the Table II below, writing in the first line the values of L, and putting in the second line the above-obtained series in such a way that the center of the series (21 in the middle) is located just under 6 (the number which we have put

TABLE II

L	—3	—2	—1	0	1	2	3	4	5	6	7	8	9	10	11	12	12	14	15
	7	3	6	9	12	15	18	20	21	21	21	20	18	15	12	9	6	3	1

aside at the beginning) of the first line. A part of the series (in this example three numbers 1,3 and 6) would then formally correspond to negative values of L. (Of course L cannot be negative in reality).

4) Invert those numbers which correspond to negative L's over the vertical line through $L = -\frac{1}{2}$, and put them in the third. line, as shown in Table III.

5) Substract the third line from the second. The resulting series. which is listed in the fourth line of Table III gives the values of Z_5 (1, 3, 5, 6, L).

TABLE III

L	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	9	12	15	18	20	21	21	21	20	18	15	12	9	6	3	1
	6	3	1													
Z	3	9	14	18	20	21	21	21	20	18	15	12	9	6	3	1

When L takes half-odd integer values, we still follow the rule that the series of numbers should be inverted with respect to the vertical line through $L = -\frac{1}{2}$, so that our scheme now appears as shown in Table IV, which represents the case of $Z_6(\frac{1}{2}, 1, 2, 2, 2, L)$.

TABLE IV

L	—7/2	—5/2	—3/2	—1/2	1/2	3/2	5/2	7/2	9/2	11/2	13/2	15/2
	1	4	9	15	21	25	25	21	15	9	4	1
					9	4	1					
Z					12	21	24	21	15	9	4	1

It can be easily recognized that this method of evaluation is a straightforward application of the relation (1.4).

Summary

Based on the relations presented in the first part of this note an asymptotic expansion (for $n \gg 1$ and $\Sigma l_i^2 \gg l_j^2$) of the angular momentum weight factor Z_n of the statistical theory of multiple particle production is worked out.

INSTITUTE OF NUCLEAR RESEARCH, POLISH ACADEMY OF SCIENCES
(INSTYTUT BADAŃ JĄDROWYCH, PAN)

REFERENCES

[1] Z. Koba, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **9** (1961), 211; I.B.J. report No, 214/VII.

[2] H. Cramer, *Mathematical methods of statistics*, Uppsala, 1945. K. Husimi, *Theory of probability and statistics*, Tokyo, 1942, [in Japanese].

[3] H. Ezawa, Nuovo Cimento, **11** (1959), 745.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX. No. 5, 1961

EXPERIMENTAL PHYSICS

Mapping of the Electric Structure of Semiconductor Surfaces by the Method of the Electron Mirror

by

E. IGRAS

Presented by L. SOSNOWSKI on February 23, 1961

Mapping of the electric or magnetic structure of a surface by means of electron optics has been the subject of a number of papers. These dealt with observation of the magnetic domains in ferromagnetics [1], [2], of ferroelectric domains in barium titanate [3] and of such electric inhomogeneities as, e.g., variations of the electric conductivity in crystals [4] and the contact potential differences between different elements of a surface presenting differences in the work function, i.e. "patch fields" [5]. The only reference to the application of this method to semiconductors is to be found in a quite short note published in 1956, wherein the possibility of obtaining an image of the p—n junction in a silicon crystal is mentioned [6]. In all the foregoing papers, an electron mirror was used for observation. The present investigation deals with the electric structure at the surface of germanium and silicon crystals, and, in particular, with the electric structure of p — n junctions in germanium formed by freezing method, electric structures at the grain boundaries in germanium and silicon, and electric inhomogeneities on the surface of silicon crystals.

The electron mirror used in the experiments consisted of a three-electrode immersion objective of two diaphragms and a cathode (Fig. 1). The crystal investigated was the cathode. The immersion objective was mounted on the ground joint of a glass tube metal-coated inside, whose other end contained the electron gun with fluorescent screen, also on a ground joint. The electrons, which were accelerated by a voltage of 5000 V (p · 10^{-4} mm.Hg.), formed a beam 1—2 mm. in diameter incident on the immersion objective. It is a well-known fact that, by an appropriate choice of the voltage on the cathode of the immersion objective, the electron beam can be decelerated and allowed to approach arbitrarily near the specimen-cathode. If the potential of the latter does not exceed that of the electron gun, the electrons will not attain the specimen but will undergo reflection at this equipotential surface, whose potential is zero as referred to that of the cathode of electron gun. Thus, by varying the difference in potential between the specimen and the electron gun cathode, the distance between the equipotential surface reflecting the electrons and the surface of the specimen can be made to vary continuously.

Now, if the surface of the specimen investigated presents electric inhomogeneities, the latter will modulate the shape of the equipotential surfaces, thus acting on the electrons as *sui generis* "micro-lenses". The "micro-lenses" modulate the density of the electrons reflected according to the size and shape of the local perturbations of the electric field near the surface. The electron image of the electric structure of the surface is magnified by the immersion objective and focussed on the fluorescent screen. It should be stressed that the electrons brought to a halt at the crystal surface and sent speeding in the opposite direction are exceedingly sensitive to the smallest perturbations of the electric field at the point where their velocity vector is reverted, as their velocity there is practically zero.

The specimens for investigation in the electron mirror were first carefully polished. Subsequent to soldering on the contacts and washing in benzene and ethyl alcohol, the specimen was placed upon the support of the mirror. These contacts served for applying appropriate bias (polarisation) voltages during the process of observation of the electric junctions. Magnifications of about 100 times were obtained on the mirror. The photographs were then optically magnified, 200 times at the most. As the photographs were taken with a usual camera at side view, images that are circular in reality present an elliptic form. The dark spot at the centre of each of the pictures corresponds to the aperture in the screen situated on the electron gun Fig. 1 shows the image of the electric structure on a germanium crystal $p-n$ junction formed by freezing method at 1 V bias voltage in the reverse direction.

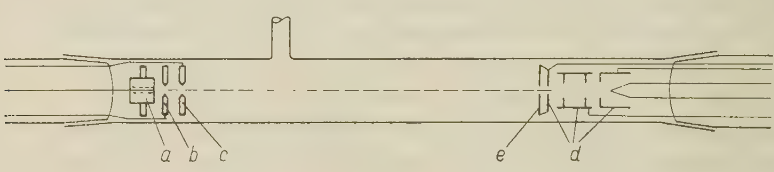

Fig. 1.

In these conditions, the junctions were sufficiently sharply visible: at lower bias voltages, the image was less contrasty, although the junction itself could still be seen at bias voltages of the order of 0.01 V. Higher voltages led to a widening of the electrically inhomogeneous region surrounding the junction; finally, at the voltage of electric breakdown, the image gradually became diffluent and vanished. If the voltage was now lowered again, the image appeared once more, however with a time lag of the order of one minute. Bias voltage applied in the direction of conduction revealed no $p-n$ junctions in the polished germanium single crystal, as now the electric field strength lines on the junction closed within the crystal. The small light spots which are to be seen on the picture may be due to micro-craters formed on the surface of the germanium crystal as a result of accidental breakdown in the neighbourhood of the cathode of the electron mirror or perhaps to electrical fields around the edge dislocations. They form microcavities with perturbations of the electric force lines all around them at the surface of the crystal, thus having a concentrating effect on the reflected electrons.

Photo. 1

Photo. 2

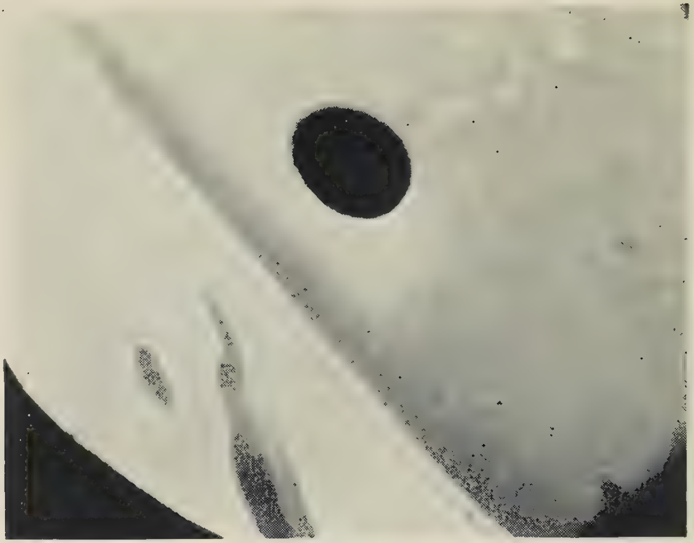

Photo. 3

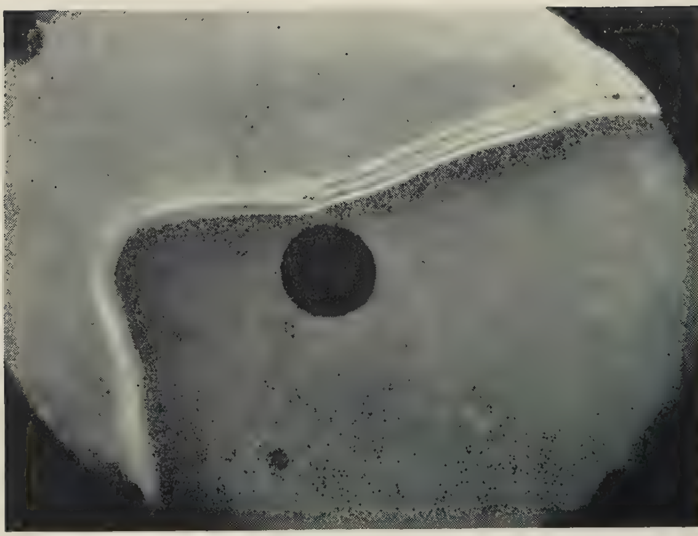

Photo. 4

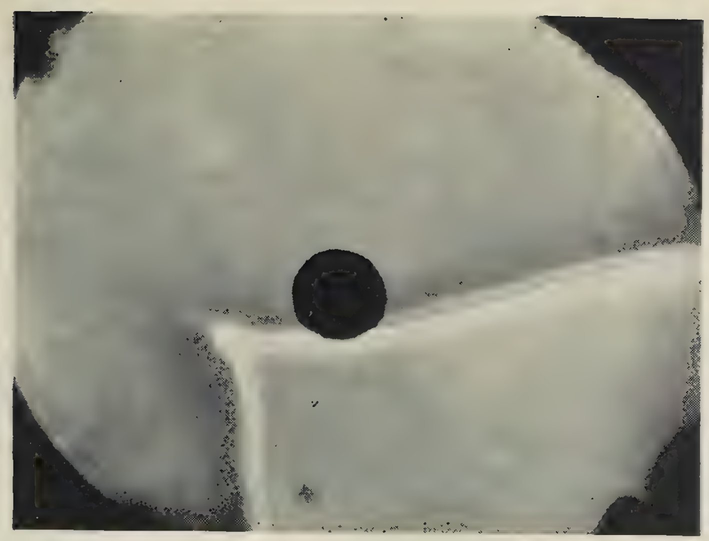

Photo. 5

Photo. 6

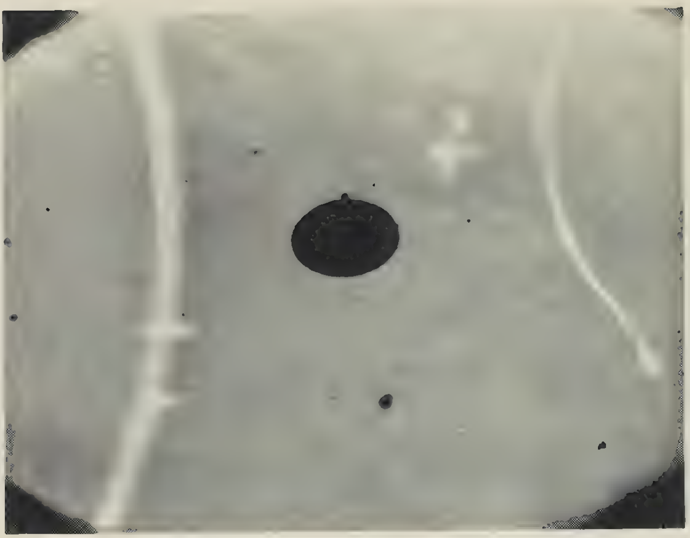

Photo. 7

Photo. 8

Changes in the electric structure on the $p-n$ junction shown in Fig. 1 were observed during illumination of the crystal with white light from a bulb. The beam concentrated by a lense was made to pass through the glass framework of the mirror onto the thin small mirror situated on the first diaphragm of the immersion objective, whence on reflection it attained the crystal. The current carriers thus generated on the $p-n$ junction made possible the observation of changes in the electric field of the barrier; in the case of a polished crystal, its image on the screen gradually became diffluent and, finally, vanished after a period of the order of 10 sec., whereas in that of a slightly etched crystal this occurred immediately. In taking the photograph, the bulb illuminating the crystal was switched off to avoid unnecessary illumination of the screen by stray light.

Photo 2 shows a $p-n$ junction, the difference with respect to Photo. 1 consisting in its having been made at the precise moment of switching on a 1 V bias. The field distribution about the junction is now characterized by the momentary appearance of two light lines separated by a dark one. The latter corresponds to a local convexity of the equipotential surface, whereas the light lines result from focussing of the electrons by potential valleys. During the several seconds ensuing, the dark line disappeared completely and only one light one remained, as in Photo 1. Such a splitting in two of the potential barrier at the moment of switching on the bias voltage was observed to occur in one of the germanium specimens only, and even in this case — not on the entire junction, but only locally. This effect is probably related to a surface layer of ions within the region of the barrier.

Photo. 3 illustrates the electric structure of the surface of a germanium crystal along the intergrowth boundary of two grains. It is usual for double junctions of the $n-p-n$ or $p-n-p$ type to exist at such boundaries. Photo. 3, indeed, shows the image of the electric field about a junction of this kind. The dark longish spot near the junction is lue to a contact potential difference at what is probably the boundary between a carbon film and the germanium surface. Such films sometimes arise in apparatus involving vacuum grease and oil pumps [7]. When the crystals were observed over longer periods in the electron mirror, such films often appeared on their surface. By [7], a 10^{-5} A electron beam will produce a carbon film 10 Å thick in a vacuum tube during 10—20 minutes. Obviously, observation with the optical microscope failed to reveal such thin films on the crystal surface, whereas in the electron mirror their presence can be observed owing to the contact potential difference between the film and the crystal.

Photo. 4 and 5 illustrate the interesting case of the electric structure at the boundary of two grains in a germanium crystal, when one is of type n and the other of type p; the junction presents the structure $n-p-n-p$. The respective pictures are those of its structure at two opposite directions of the bias. At a first glance, the structure of the same boundary in either picture would not seem to be the same. The differences, however, are easily explained by taking into account the difference in the angle of inclination of the potential relief over the barrier with respect to the plane of the crystal in either case. The value of the angle is clearly dependent on that of the component of the electric field strength due to the bias voltage, nor-

mal to the barrier. For this reason, the field over the $n-p-n-p$ barrier in the pictures not only exhibits an apparent shift, but also differs in shape. Characteristically, pictures of barriers on semiconductor surfaces show them to be forming boundaries between the regions of stronger and weaker illumination. This is due to the distribution of the equipotential surfaces on either side of the barrier. The electrons reflected from the portion of the equipotential surface more distant from the crystal surface are less effective in illuminating the screen, since they undergo greater absorption by the diaphragm of the immersion objective than those reflected from the remaining portion of that surface. For a given direction of bias voltage (polarisation), the angle of inclination of the potential relief over the barrier increases simultaneously with the bias. This was clearly seen in the electron images. E.g., Photo. 6 shows the image of the electric field of the same $n-p-n-p$ barrier at a relatively high bias voltage approaching that of breakdown (55 V). The image of the barrier field has begun to become diffluent and to disappear, and single lines are no longer to be seen. On the other hand, the shift of the barrier with respect to other details of the image picture is clearly visible, such as e.g. the position of point "1" which is independent of the bias voltage and remained unchanged in the electron image. It should be clearly stated once more that this is no shift of the barrier on the crystal surface, but solely an effect of focussing the barrier by the immersion objective at different points of the screen due to the different angles of inclination of the barrier with respect to the crystal surface.

Electric inhomogeneities on a silicon crystal surface probably due to a strong local concentration of defects is shown in Photo. 7, which was taken at a bias of 0.5 V. The latter, when applied to the same crystal in the opposite direction, caused diffluence of the charges on the defects, as seen in Photo 8.

Owing to the electron mirror, the existence and type of electric micro-inhomogeneities on semiconductor crystal surfaces can be ascertained rapidly and with relative precision. Moreover, their size and shape can be established, and the dynamics of their changes due to illumination, heating or an electric field observed directly. For the time being however, the data thus collected are but qualitative. In the case of $p-n$ barriers, the quality of the mapping improves as the bias voltage decreases. As already stated, a high bias will bring about an apparent shift of the barrier. However, at bias voltages of less than 0.5 V the apparent shift of the barrier is practically unnoticeable, the image on the screen remaining sufficiently contrasty.

It is to be expected that fuller information on the nature and properties of the electric inhomogeneities on the semiconductor crystals investigated will be obtained by employing the foregoing method in conjunction with micro-X-ray and electric investigation. The electron mirror provides for a mapping of the electric structure near the surface only, together with its dynamics, in a manner that is highly spectacular and direct though qualitative in principle. The foregoing investigation of the electric fields at the surfaces of semiconductors by the method of the electron mirror are of a preliminary character, and research thereon is continuing.

The author wishes to thank Professor L. Sosnowski for his valuable advice and for his interest throughout the present investigation.

INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT FIZYKI, PAN)

REFERENCES

[1] G. V. Spivak, DAN SSSR, **105** (1955), 965.
[2] L. Mayer, J. Appl. Phys., **28** (1957), 975.
[3] G. V. Spivak, E. Igras, I. A. Pryamkova, Kristallografia, **4** (1959), 123.
[4] L. Mayer, J. Appl. Phys., **26** (1955), 1228.
[5] G. V. Spivak, I. A. Pryamkova, V. N. Lepeshinskaya, DAN SSSR, **130** (1960), 751
[6] G. Bartz, G. Weissenberg, Naturwissenschaften, **44** (1957), 229.
[7] R. W. Christy, J. Appl. Phys., **31** (1960), 1680.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 5, 1961

EXPERIMENTAL PHYSICS

Effect of Cathodic Hydrogen on the Magnetic Properties of Thin Electrolytic Nickel Films

by

L. KOZŁOWSKI and S. KUBIAK

Presented by M. MIĘSOWICZ on February 27, 1961

Introduction

In research work carried out hitherto on the effect of cathodic hydrogen on the magnetic properties of ferromagnetics [1]—[3], the sample investigated was 0.1 to 0.3 mm. thick. Investigation dealt essentially with the changes in the coercive force H_c occurring in the process of cathodic charging of the sample with hydrogen, because changes in magnetization were not accessible to observation owing to the fact that adsorption of hydrogen is a surface phenomenon.

Baranowski and Śmiałowski [4] found nickel films 0.7 to 30 μ thick, deposited electrolytically on copper, to become saturated with hydrogen to a concentration of 130 cm^3. per 1 gm. Ni in the process of cathodic charging. This corresponds to an atomic ratio of 0.7 hydrogen atoms per atom of nickel. If the thickness is greater, the hydrogen accumulates chiefly within a surface layer of constant thickness 30 μ.

Investigation of the structure of nickel films charged with cathodic hydrogen [5]—[7] revealed, in addition to the normal line of nickel, others giving indication of an increase in the space lattice constant (amounting to about 6%) without change of lattice type. The new H/Ni phase, however, is instable and vanishes after several hours of desorption.

Similar phenomena occur in the Pd-H system. When palladium is saturated with hydrogen, a Pd/H phase of increased lattice constant also arises. The latter is relatively stabile, and its paramagnetic susceptibility tends to zero as the holes in the 4*d* band are filled up with electrons from the hydrogen atoms adsorbed. The 3*d* band of nickel presents, on the average, 0.6 electron hole which can be filled up by electrons from hydrogen at a atomic ratio of H/Ni = 0.6. The H/Ni phase of this concentration should possess a considerably reduced saturation magnetization.

It was the aim of the present authors to investigate the saturation magnetization of nickel films charged with cathodic hydrogen according to the method proposed by Baranowski and Śmiałowski [4].

Preparation of samples

The nickel films were deposited on small copper tubes of outer diameter 5 mm. and inner diameter 3 mm., having a length of 130 mm. The following electrolyte was used:

$NiSO_4 \cdot H_2O$	140 gm./l.
$NaSO_4 \cdot 10H_2O$	50 gm./l.
$MgSO_4 \cdot 10H_2O$	30 gm./l.
H_3BO_3	20 gm./l.
NaCl	5 gm./l.

The value of *pH* was kept within the limits of 5—5.5; the temperature of the electrolyte was 18°C, and the current density — 5 mA./cm.^{2}. As far as possible, conditions were maintained the same in all cases. The thickness was determined gravimetrically.

Previous to cathodic polarization, the samples were annealed at 400 C and about 10^{-2} mm.Hg. for 2 hours in a vacuum furnace and then cooled with the latter during 3 hours until a temperature of 120°C was reached, then the current and vacuum were switched off and the samples together with the furnace were left to cool down to room temperature.

The samples thus prepared were charged with cathodic hydrogen at room temperature in a normal solution of H_2SO_4 with an addition of 2 gm./l. thiourea. The current density of the polarization process was 20 mA./cm.^{2}. The anode consisted of a platinum wire encircling the sample spirally.

Results of measurements

The electrolyzer with the sample was placed within the coil of a Bozorth type astatic magnetometer made by one of the authors [8]. Fig. 1 shows a diagram of

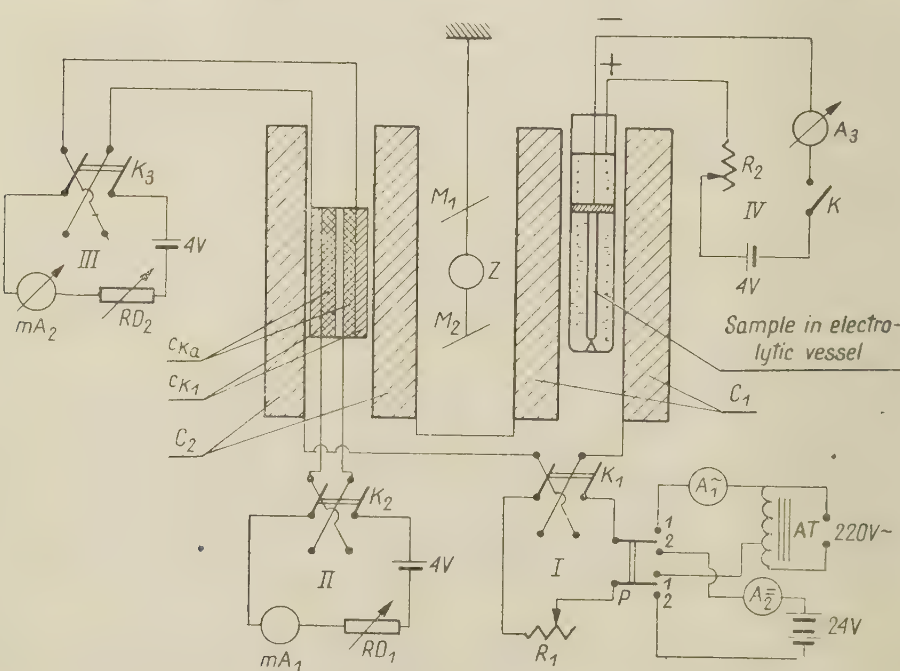

Fig. 1. Diagram of measuring circuit

the measuring circuit. Circuit I served for feeding the magnetizing coils C_1 and C_2 with DC current or for demagnetizing the samples with AC current, according to the position of the switch P. Circuits II and III provided the energy for the mea-

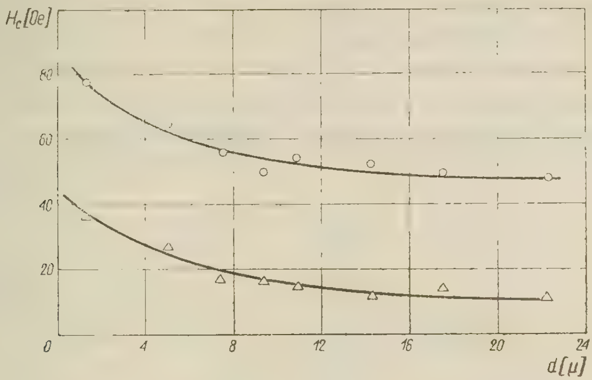

Fig. 2. H_c plotted against thickness of nickel film
o—o—o— before annealing, △—△—△— after annealing

suring coils C_{k1} and C_{k2} compensating the magnetization and the variations of the magnetization of the sample, respectively. Circuit IV served for cathodic charging of the sample investigated.

Preliminary measurements of the magnetization at 200 Oe magnetic field intensity (I_{200}) and magnetic remanence I_r showed these quantities to remain constant

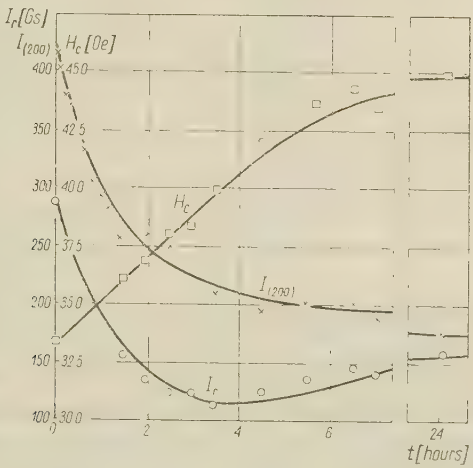

Fig. 3. I_{200}, I_r and H_c as a function of the time of cathodic charging of a nickel film 5 μ thick

in samples before and after annealing throughout the 1 μ to 24 μ range of thickness of the nickel films. Fig. 2 illustrates the dependence of the coercive force H_c on the thickness of the nickel layer before and after annealing. Fig. 3 brings the values

of I_{200}, I_r and H_c *versus* the time (in hours) of cathodic charging for a nickel layer 5 μ thick. H_c is seen to increase, whereas I_{200} decreases strongly as the charging with hydrogen proceeds.

In order to establish the quantitative dependence of the magnetization I_{200} on the content of cathodic hydrogen, the variations of I_{200} *versus* the amount of hydrogen desorbed were investigated in detail. Two nickel layers, 7.5 μ and 14.1 μ thick, were charged cathodically during 20 hours, rinsed in water and placed within the volumenometer shown in Fig. 4. The volumenometer with the nickel film was

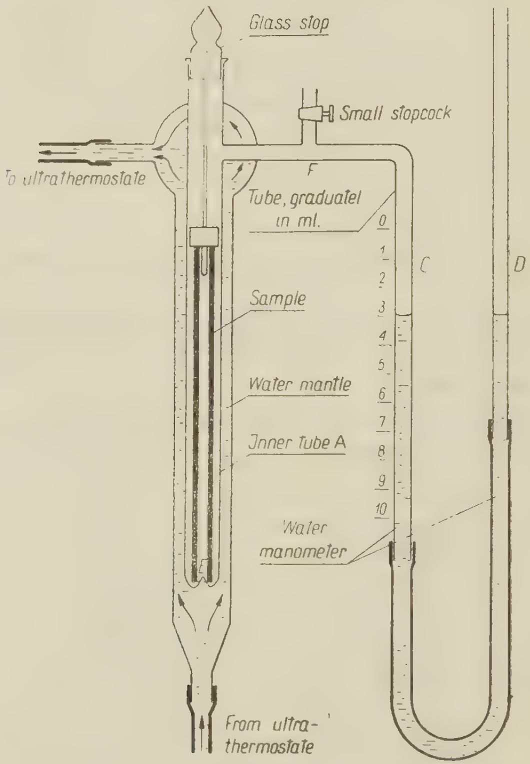

Fig. 4. Volumenometer for measuring amount of hydrogen desorbed

placed in the coil of the astatic magnetometer (in place of the electrolyzer), and I_{200} and the volume V_H of the hydrogen desorbed were measured *versus* the time of desorption. The graphs in Fig. 5a show the variations of I_{200}, whereas those of Fig. 5b illustrate the variations of V_H as *versus* the time of desorption of the hydrogen from the 14.1 μ nickel film. After the first polarization, I_{200} fell from 500 to 200 gauss. During 250 minutes of desorption, the value of I_{200} does not vary, and the amount of hydrogen desorbed is exceedingly small. After a second charging I_{200} varies insignificantly, increasing somewhat as hydrogen is desorbed from the layer. Similar graph forms were obtained subsequent to a third and fourth charging

process. Almost reversible and reproducible variations resulted after a fifth and sixth charging. Previous to desorption, I_{200} had decreased almost to zero, whereas during desorption I_{200} increases to a value of 175 gauss, hydrogen desorbed amounting to 30 cm.³. Subsequent to the first charging with hydrogen — as shown in Fig. 5c — there is an obvious decrease in I_{200}; however, no hydrogen leaves the

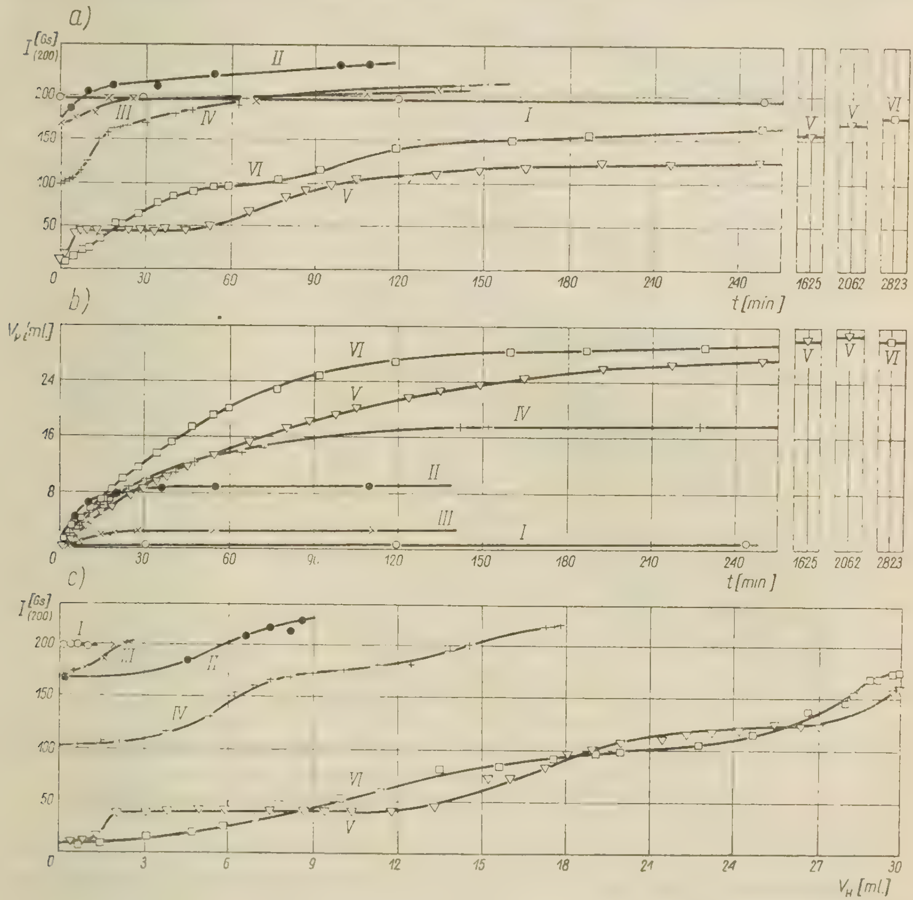

Fig. 5. Variations of I_{200} and V_H *versus* the time of hydrogen desorption (a, b) and of I_{200} *versus* the volume of hydrogen desorbed (c). Nickel film 14.1 μ thick

layer. On the other hand, after repeating the charging process six times, a situation is achieved where I_{200} varies from about 10 to 175 gauss, and hydrogen desorbed amounts to 30 cm³. each time. A first charging with cathodic hydrogen will produce irreversible changes in I_{200}.

Fig. 6 shows similar relationships for the nickel layer 7.5 μ thick. In the first charging process, magnetization decreased from 500 to 180 gauss. On the other hand, the fifth and sixth polarization yielded reversible changes of the magnetization from 2.5 to 135 gauss with 15 cm³. hydrogen desorbed.

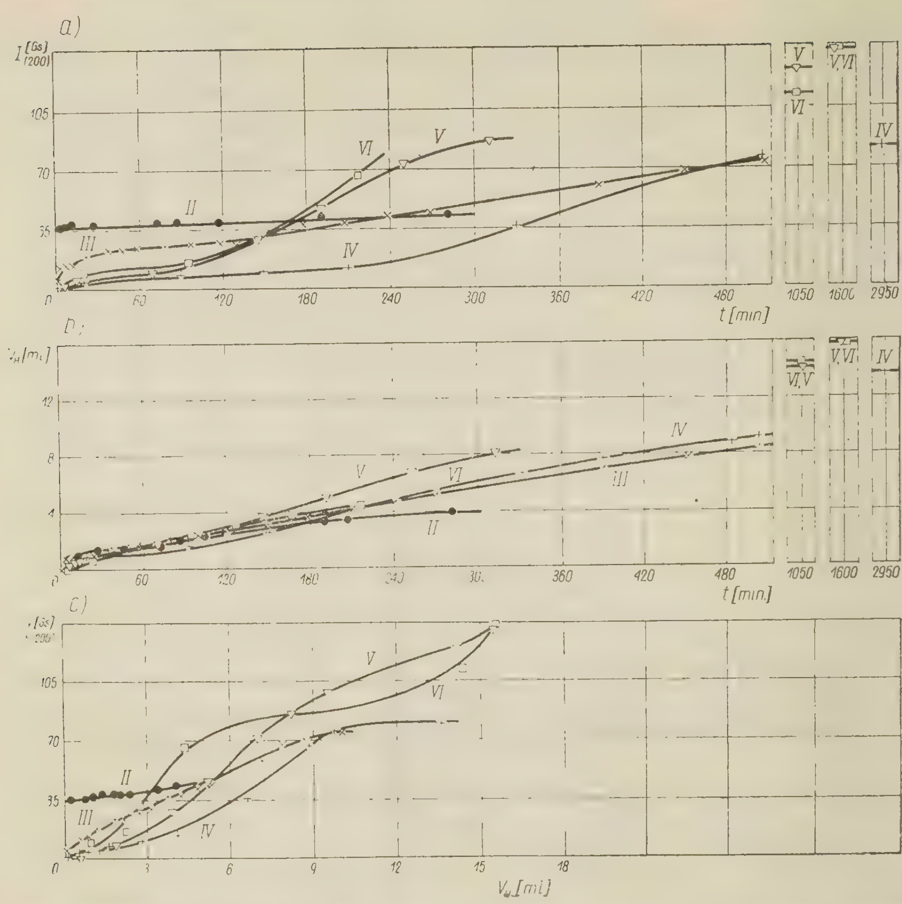

Fig. 6. Variations of I_{200} and V_H with the time of hydrogen desorption (a, b) and of I_{200} with the volume of hydrogen desorbed (c). Nickel film 7.5 μ thick

In order to establish the character of the irreversible component of the change in magnetization, the quantity I (magnetization), was measured in either layer subsequent to desorption as a function of the magnetizing field strength, in the Weiss type electromagnet, using the balistic method. The results are shown in Fig. 7, wherein the values of $1/H \cdot 10^3$ (Oe^{-1}) are plotted on the axis of abscissae and the corresponding values of I on that of ordinates. Charging with cathodic hydrogen was found to produce such considerable hardening that the layer failed to become saturated in a field of 200 Oe. However, at a magnetic field intensity of 5000 Oe, magnetization amounts to as much as 480—490 gauss, which is the value of I_s in annealed nickel film previous to cathodic charging.

In order to obtain complete insight into the chages accompanying cathodic charging of nickel films the 14.1 μ layer was polarized succesively during 20 hours and the time-dependent changes in I_s were measured in a DC field of 8000 Oe simul-

taneously with the amount of hydrogen desorbed. The results obtained in three successive processes of cathodic charging are plotted in Fig. 8. Notwithstanding the differences in the initial I_s values immediately after polarization, the reversible component of the changes in I_s related to the hydrogen desorption could be established with certainty. The latter changes vanish after desorption has continued for a sufficiently long time.

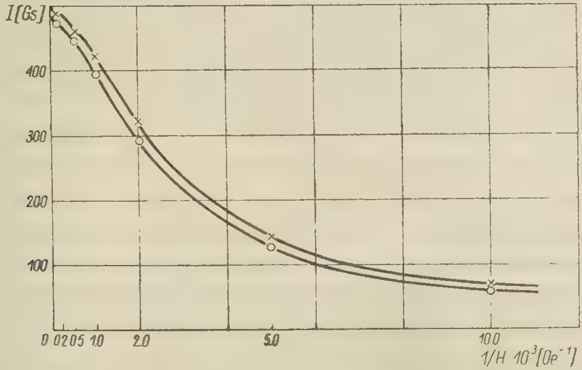

Fig. 7. Variations of I *versus* $1/H \cdot 10^3$ (Oe^{-1}) in nickel films after hydrogen desorption x—x—x 14.1 μ layer thickness, o—o—o 7.5 μ layer thickness

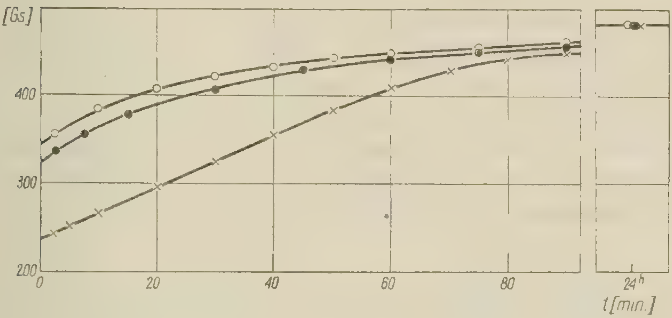

Fig. 8. Variation of I_s with the time of hydrogen desorption from a nickel film of thickness 14.1 μ. Magnetic field intensity 8000 Oe

Discussion of results

From the volume of hydrogen desorbed and from the mass of the electolytically deposited nickel layer, the atomic ratio of hydrogen per Ni atom is found to be 0.55 H/Ni on the average.

Since the 3*d* band of nickel is believed to have an average of 0.6 electron holes per Ni atom, the volumetric results relating to reversibly desorbed hydrogen seem to confirm the conclusions of Baranowski and Śmiałowski concerning the filling up of the holes of the *d* band in nickel by electrons taken from the hydrogen atoms.

The number of Bohr magnetons per Ni atom was computed for either annealed film and a mean value of 0.61 was obtained. The latter figure differs but insignificantly from that of 0.6, which is considered to correspond to the bulk metal. From Fig. 8, the magnetization of saturation in the 14.1 μ layer measured in a magnetic field of 8000 Oe amounts to 484 gauss, whereas after 20 hours of cathodic charging it decreases to 330, 240 and 350 gauss. The magnetic moments of the Ni atom and the H/Ni phase are 0.58, 0.40, 0.29 and 0.42 Bohr magnetons, respectively. The maximum per cent change in the magnetic moment of nickel atom related to adsorption of a hydrogen atom attains 50%.

Thus, hydrogen atoms adsorbed by the nickel layer produce two effects:

a) internal stress leading to an increase in H_c and magnetic hardening, and

(b) a decrease by as much as 50% of the saturation magnetization of the nickel layer.

Adsorption of hydrogen to the amount of 130 cm^3./1 gm. Ni fails to reduce magnetization of saturation to zero as would seem natural on the assumption that the electrons from hydrogen fill up the electron holes of the 3*d* band of nickel.

The foregoing results are in contrast to those obtained when dissolving hydrogen in palladium, where the hydrogen is thought to occur as proton having given off its electron to the 4*d* band. Hence, in the case of nickel, we are dealing with a process that is not purely ionic but consists of filling up by the electrons in part the 3*d* and in part 4*s* bands.

During the initial cycles of cathodic charging the hydrogen entering the micropores and other irregularities of the lattice produces considerable internal stresses which modify the magnetic hardness of the nickel layer. Such stresses, in turn, involve the increase of H_c of the layer exemplified in Fig. 3.

Subsequent to preparation of the layer by the method of Baranowski and Śmiałowski, adsorption of hydrogen occurs with the formation of a H/Ni phase of reduced saturation magnetization, an instable phase which vanishes after several hours desorption, the one observed by Janko [5]—[7] in the course of his structural investigation.

Dietz and Selwood [9] found similar changes in magnetization to occur when hydrogen is adsorbed on nickel-kieselguhr hydrogenation catalysts wherein the nickel particles had a diameter ranging from 10 to 100 Å. The magnetization of the catalyst previous and subsequent to saturation with hydrogen was measured at 4.2°K. According to the method of preparation of the catalyst Dietz and Selwood found the number of electrons filling up the 3*d* band of nickel per 1 atom of hydrogen adsorbed to vary within the limits of 0.39 and 0.72.

The authors wish to thank Professor M. Śmiałowski and Docent Z. Szklarska-Śmiałowska for their valuable hints concerning the preparation of the samples and measurement of the hydrogen desorbed, and for their numerous helpful discussions of the results.

DEPARTMENT OF PHYSICS, NORMAL SCHOOL, KATOWICE
(KATEDRA FIZYKI, WYŻSZA SZKOŁA PEDAGOGICZNA, KATOWICE)

REFERENCES

[1] L. Kozłowski, Archiwum Hutnictwa, **2** (1957), 223.
[2] — , Bull. Acad. Polon. Sci., Cl. III, **5** (1957), 519, 733.
[3] L. Kozłowski, A. Gucwa, Zeszyty Naukowe, Sekcja Fizyki WSP, **2** (1960), 5.
[4] B. Baranowski, M. Śmiałowski, Bull. Acad. Polon. Sci., Sér. sci. chim., géol. et géogr., **7** (1959), 663.
[5] A. Janko, Naturwissenschaften, **47** (1960), 225.
[6] — , Bull. Acad. Polon. Sci., Sér. sci. chim., **8** (1960), 131.
[7] A. Janko, P. Michel, Comptes rendus, Paris, **251** (1960), 1001.
[8] S. Kubiak, Zeszyty Naukowe, Sekcja Fizyki, WSP, **2** (1960), 21.
[9] R. Dietz, P. Selwood, J. Appl. Phys., **30** (1960), 101 S.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 5, 1961

Measurements of Thermoelectric Power in InSb

by

J. GINTER and W. SZYMAŃSKA

Presented by L. SOSNOWSKI on March 11, 1961

Introduction

Exact measurements of the thermoelectric power in semiconductors are of considerable importance for the investigation of their fundamental properties, especially for the determination of the band structure and scattering mechanism. Since no systematic data on the dependence of thermoelectric power in InSb on the temperature for various carrier concentrations in the range of extrinsic conduction are to be found in the literature published hitherto, it seemed worth while to carry out such measurements.

Experimental method

Measurements of the thermoelectric power in materials of high thermal conductivity such as indium antimonide present particularly great experimental difficulties. Conditions must be provided to eliminate the systematic error due to the thermal resistances between specimen and thermocouple.

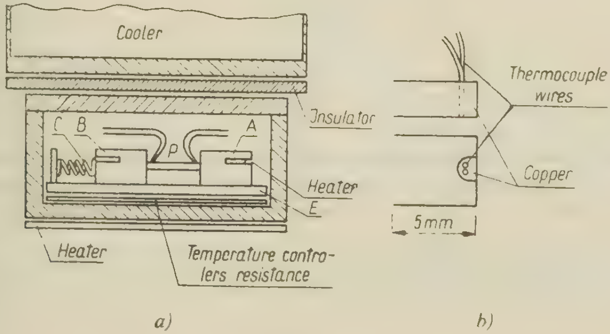

Fig. 1.

The present authors attempted to eliminate this difficulty as follows: the terminals of the thermocouples were attached to the specimen directly by means of a copper layer deposited electrolytically. The position of the thermocouple wires (copper, constantan) is shown in Fig. 1 b. The cop-

per wires of the thermocouples acted at the same time as reference electrodes for the thermal EMF. Typical apparatus was used in the measurements (shown schematically in Fig. 2). The specimen P was placed between two copper blocks A and B. Both blocks were attached to the support E, which was made of an insulating material, the one (A) being immobilized and the other (B) moving freely on rails. The spring C provided for constant good contact between the blocks and the specimen. In order to reduce heat exchange by convection, the specimen was wrapped in cotton-wool. The blocks and specimen were placed within a brass container with thick walls connected through an insulating layer with a thin-walled brass vessel containing a refrigerating substance (solid CO_2 or liquid nitrogen). The entire apparatus was placed within a Dewar flask.

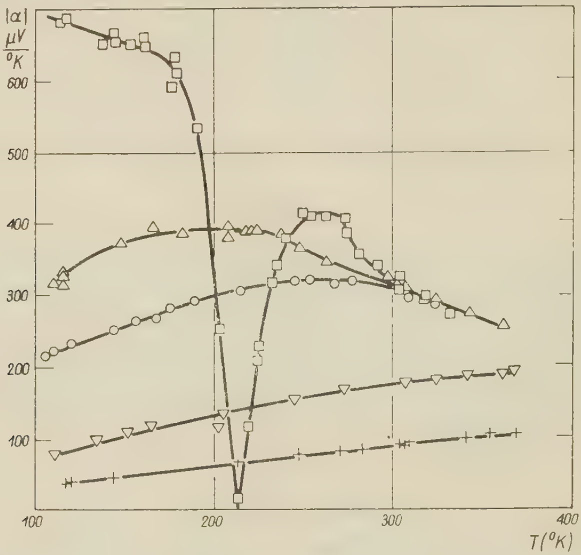

Fig. 2.

The difference in temperature across the sample was produced by heating one of the blocks by means of a small heater situated within it. During the measurement, this difference amounted to several degrees. The measurements were carried out at steady state conditions (temperature variations did not exceed 0.1°/min.). The temperature was regulated by means of a temperature controller acting on the principle of bridge connection. The mean temperature of the sample was varied throughout the range of 100—400°K. The thermal EMF was measured with the accuracy of 1 μV by means of a compensator.

The divergence between the results obtained for specimen No. 4 at room temperature by means of the foregoing apparatus and the one proposed by Tauc et al. [1] lies within the limits of spread of experimental results. Hence, it may be concluded that the experimental results are free of systematic error.

The spread amounted to $\pm 1.5\%$ at room temperature and $\pm 2.5\%$ at about 100°K. The authors feel that a higher degree of accuracy could be attained by using thermocouple wires and InSb specimens of higher homogeneity.

Results

The results of the measurements are given in Fig. 2 and in the Table for two different temperatures. The carrier concentration in the impurity range for the respective specimens has been computed from measurements of the Hall coefficient at the temperature of liquid nitrogen.

TABLE

No.	Type	H Gs	R cm^3/C	Concentration (cm^{-3})	α (120°K) μV/°K)	α (300°K) μV/°K
1	*n*	3900	6.5	9.6×10^{17}	—39	—88
2	*n*	2200	53	1.25×10^{17}	—84	—173
3	*n*	2200	430	2.0×10^{16}	—232	—304
4	*n*	1400	1870	5.0×10^{15}	—340	—318
5	*p*	1400	4060	1.5 10^{15}	+675	—325

For the *n* type specimens, the carrier concentration was computed from the results of J. Kołodziejczak [2] on the assumption of scattering on ionized impurities. The $R(n)$ dependence as computed from that paper is shown in Fig. 3. In the case of the *p* type specimen, because no reliable data on the scattering mechanism were

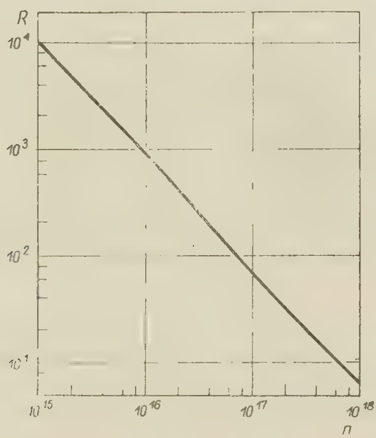

Fig. 3.

available, the evaluation was made according to the formula $p = 1/Re$. In order to make a detailed analysis of the obtained results it is necessary to carry out systematic measurements of the Hall coefficient and conductivity dependence on the temperature and the magnetic field strength. This analysis will be published later.

The authors are indebted to Professor L. Sosnowski for his interest in the present investigation, and to Mgr W. Giriat for preparing the crystals.

INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT FIZYKI, PAN)

REFERENCES

[1] J. Tauc, J. Bednar, A. Abraham, Chechosl. Journ. Phys., **3** (1953), 314.
[2] J. Kołodziejczak, Acta Phys. Polon., [in press].

BULLETIN DE L'ACADEMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. IX, No. 5, 1961

GEOPHYSICS

A Dislocation Theory of the Earthquake Processes

by

R. TEISSEYRE

Presented by L. INFELD on March 16, 1961

The dislocation theory of elastic media is applied to the problems of seismic activity mechanism and to the conditions of internal energy release in the Earth. The physical dislocations join the interaction of shear stress field with medium's inhomogeneities in the unique theory. The possibility of grouping dislocations in various combinations closely approaching the real deformation and stress conditions in the medium suggest to assume a physical dislocation as an elementary dynamical unit. From this point of view the equivalence of the field of the appropriate dislocation series with the crack field [8] arguments in favour of using the dislocation theory rather than the crack theory. Also A. V. Vviedenskaia [6], [7] introduced the physical dislocations as an essential element of the mechanisms of quakes and as basis for dynamic modelling of the foci.

In our former papers many attempts were devoted to the problem of dislocation sources in an elastic medium [1]—[3], [5]. Inhomogeneities and intrusions contained in the medium speak in favour of the process of synthesis and grouping of the small loop dislocations. Structural dislocation sources in the regions of great discontinuity zones are discussed in [5]. The problem, however, is not explained satisfactorily.

In the following it will be assumed — independently of the question relating to mechanism of the sources — that there exists in every medium a certain dislocation density which depends on the degree of inhomogeneity, the history of the body and also on the value of the external stresses acting on the given body.

In the case of the earth's crust and mantle we shall assume the existence of a dislocation field with different displacements $\dot{b}$, and with a dislocation density n depending, a.o., on the level of stresses in the medium.

Without entering here upon a closer examination of the essential nature of the earthquake mechanism, we may assume that the number of earthquakes N within a given time interval is proportional to the value n.

The dislocation theory leads to the following relation

$$NE^2 = \varrho^{-1} \text{ const}, \tag{1}$$

where N is the number of earthquakes with a given energy E, and ϱ — is the linear dimension of a dislocation.

The dimension ϱ depends on the structural systems characteristic for the given area — one may in rough approximation assume $\varrho = \text{const}$. From this we obtain the constancy of the product NE^2.

On the other hand, the research and observational data regarding energy E, magnitude M and earthquake frequency N show the following correlations [9], [10]:

$$\begin{cases} \log E = a + bM, \\ \log N = g - fM. \end{cases} \tag{2}$$

The values for b are lying within the limits 1.6—2.4. Values g depends on the seismic region under investigation and the depth of the observed quakes. Value f is nearly constant and lies within the range 0.8—1.2. Putting $\frac{b}{f} \approx 2$ we obtain from (2) the relation

$$\log NE^2 = a + 2g \tag{3}$$

expressing the experimental law of the approximate constancy of product NE^2.

The above considerations upon the earthquake statistics are independent of the quake mechanism.

The following categories of earthquakes can be distinguished.

a. Quakes connected with violent formation or movement of dislocations; this mechanism corresponds in principle to the classic models of quakes [6], [11], [12]. In [1] and [2] we have pointed out that this mechanism leads to only partial release of the internal energy, whereas its main part is being converted into energy of the dislocation formed.

b. Earthquakes connected with the approach of the dislocation to the earth's surface [1], [2], [4].

c. Earthquakes connected with the approach of the dislocation to the internal discontinuity surface with lower rigidity module μ [4]; a part of the dislocation energy, proportional to the jump of $\Delta\mu$, is released.

d. Quakes resulting from mutual annihilation of two dislocations with opposite sign at their junction [1].

All the enumerated categories of quakes can be represented by the mechanism of formation or vanishing of a contour dislocation or a pair of dislocations. This applies even to the movement of the dislocation which can be represented as successive formation of dislocation pairs [13]. The same applies in the case of a dislocation approaching the discontinuity surface: the pair is here formed by the dislocation appoaching the boundary surface and by the image dislocation [4]. The model representation of these processes corresponds to that of dipole pairs with moments, which is stressed in the papers of A. V. Vviedenskaia and H. Honda. The potential energy of a pair of dislocations situated at the distance L is [15]

$$\begin{cases} E = \dfrac{\mu b^2 l}{2\pi(1-\sigma)}\left(\ln\dfrac{L}{r_0} - \dfrac{1}{2}\right) \text{(edge dislocation)} \\ E = \dfrac{\mu b^2 l}{2\pi} \ln\dfrac{L}{r_0} \text{ (screw dislocation),} \end{cases} \tag{4}$$

where l — length of dislocation, r_0 — radius of dislocation line, b — value of displacement vector. If two dislocations approach each other, in consequence of their interaction and under the influence of the external field p, very rapid acceleration of the approach movement takes place at a certain mutual distance L and junction occurs, causing total or partial annihilation. The energy (4) can be regarded as the total energy released in deformation work and seismic radiation.

The expression for the energy of a dislocation or a pair of dislocations contains the term r_0, i.e. the radius of the dislocation line. The continuous elastic medium is cylindrically hollowed along the dislocation line whereby the singularities are removed. The equilibrium radius r_0 of the cylinder can be found from the equivalence condition between the dislocation energy and the surface energy of the cylinder formed [15].

In the case of a dislocation in the earth's interior, the formation of a hole (in the proper meaning of the word) is rather dubious in view of the high confining pressures. It should be assumed that the material within this surface is being strongly deformed by extensive crushing and disintegration. The radius r_0 depends on the strength of the material through the amount of the surface energy.

The movement of the dislocation line (front) causing extension of the dislocation area, leads in consequence also to extension of the deformations, created by the high stresses within these energetic surfaces. The occurrence of changes in the structure of rock masses along dislocational planes is being regularly observed. Such layers surrounding the plane of a dislocation are termed in geology mylonite or ultramylonite layers, depending on the degree of material disintegration.

We can now approximately segregate the part representing work from that representing radiation in the formulas for total energy of an earthquake (4). Let us imagine that a violent movement of the dislocations $\perp$ and $\top$ begins at the distance L. The main part of their mutual energy goes at the beginning into deformation work, but with rising velocity of the movement the radiation losses increase. Now, by neglecting the radiation generated by the movement of the dislocation, we can make the following estimation. In view of the presence of energetic surfaces around the dislocation lines $\perp$ and $\top$, contact between the dislocations will not occur at a distance equal to zero, but at $2\,r_0$. The values of the mutual energy at a distance of $2\,r_0$ can, as a first approximation, be considered as the main part of the radiation energy, as this part is emitted in the annihilation process. The remaining part of the energy has been transformed into deformation work on the way from distance L to $2\,r_0$. The radiation losses during the movement are, as mentioned above, neglected here.

Using the formulas (4) we obtain expressions for the radiation energy:

$$
(5)\qquad \begin{cases} E_s = \dfrac{\mu b^2 l}{2\pi(1-\sigma)}\left(\ln 2 - \dfrac{1}{2}\right) \text{ (edge dislocation)} \\[2ex] E_s = \dfrac{\mu b^2 l}{2\pi} \ln 2 \text{ (screw dislocation)}. \end{cases}
$$

Formulas for the deformation work E_d are given by: $E_d = E - E_s$. The displacement fields for dislocation movements and for energy release processes are given in paper [5].

A field of screw dislocation reaching the Earth's surface is expressed by [5]

$$u_x^0 = -\frac{b}{\pi} \operatorname{arc\,tg} \frac{y}{1.5\,H}\,. \tag{6}$$

This result and the previous consideration on the energy value may be compared with the results of P. Byerly and I. De Noyer [16]. In [16] a formula for the deformation work during an earthquake is presented.

$$E_d = \frac{\mu\, H b^2\, l a}{4\pi}\,. \tag{7}$$

(in the present paper we write H, l, b instead of D, $2L$, $2M$ [16]). Formula (7) is based on the principles of the elastic rebound theory [17] which are generally accepted for dynamic processes in the earth.

Formula (7) contains the value a, which is the parameter in the $u_x = -\frac{b}{\pi} \operatorname{arctg}(a\,y)$ function, chosen empirically to the geodetically measured horizontal displacements in the function of their distance from the fracture plane. Now it is just this empirically selected arctg function which corresponds exactly to the field of the displacement of a screw dislocation reaching the earth's surface (6). This is a very significant confirmation of the dislocation theory of earthquakes [18]. Here one has to take into account that the field of displacement of a screw dislocation near the earth's surface is given by the sum of the dislocation field and its image.

Comparison of Byerly and DeNoyer's empiric formula with the displacement field of a pair dislocations, corresponding to the depth of the first impulse H, leads to the equality $a = \frac{1}{1.5\,H}$.

Whence, a direct method of computing the depth of quakes from the geodetic data. The values thus obtained for H are slightly lower than the depths estimated in [16]. Formula (7) for deformation work is now simplified to the expression:

$$E_d = \frac{\mu b^2\, l}{6\pi}\,. \tag{8}$$

The essential factors of this formula agree with previous ones. More detailed discussion of the earthquake mechanism requires a study on the movement of dislocations.

The displacement of a dislocation is connected with the cracking of material at the dislocation front, whereupon occurs relative shifting along the surface of the crack, equal to vector b and, finally, juncture of the surfaces. Our assumption here has been that the material in the earth is fractured by a shearing process. A. Griffith assumes in his theory of cracks [19], [20] (which closely approaches our

present considerations) that cracking of bodies is caused by exceeding of the tensile strength at the front of the crack, which thus is enlarged. This results from the characteristic field distribution of the stresses. But, in view of the great confining pressures in the earth's interior, one may rather presume that fracturing is the result of a shearing process, similarly as in Mohr's theory [21], [22]; this becomes apparent primarily from the increase of tensile strength at high confining pressures [23]. In the approach to the earth's surface the situation may, however, become reversed in favour of the tensile mechanism.

In the case of a screw dislocation, the shearing character of the field is decisive for the extension of the dislocation.

The analysis of the coefficients in equation of motion and the analysis of the forces exerted on dislocation leads to the following formula [5]

$$\mu b \over 2\pi c^2}\cdot\frac{\dot{v}}{\lambda_0}+\nu\frac{v}{\lambda_0}+S=p, \tag{9}$$

where v — dislocation velocity, ν — viscosity coefficient, S — shearing strength, c — wave velocity, p — shear stress, λ_0 — length unit in the direction of the dislocation movement.

On the basis of this formula of equation of motion the elementary earthquake replica theory is founded. Let us take the series ($n \geqslant 2$) of the dislocation $\perp_1$, $\perp_2, \ldots, \perp_n$, the co-ordinates of the dislocation lines in the plane, perpendicular to the dislocation surface, will correspondingly be denoted by $z_1, z_2, \ldots, z_n$. We shall consider the case of dislocation with equal sign, whose interaction has repulsive character. The constant external field pushes the dislocations in the direction of the locked dislocations, in consequence of which a certain state of equilibrium in the system of interior stresses is formed [8]. In result, a very strong internal field acts on the locked dislocation $\perp_1$.

The multiplied field acting on $\perp_1$ can cause a movement of this dislocation, and subsequently the release of its energy at the earth's surface. In this case, the role of leading dislocation is taken by the dislocation $\perp_2$, whose movement will be determined by the new distribution of the stress field. This field can, by shifting the dislocation $\perp_2$ to the boundary surface, cause the release of its energy. In this way occurs the secondary quake, called replica. Subsequently, a similar process may cause a quake due to the $\perp_3$ dislocation reaching the surface.

A detailed calculation lead to the relation between replica times T_i, and their displacement vector b_i.

$$T_i=\frac{\alpha k_{i-1}}{\sigma_i^0(\sigma_i-\sigma_i^0)}\ln\frac{\sigma_i-\gamma}{\sigma_i^0-\gamma}, \tag{10}$$

where $\gamma=\frac{S}{p}$ is the ratio of static shearing strength to the value of the field of shearing stresses — the movement is conditional on $\gamma<\sigma_i^0$; $\alpha=\frac{\nu}{p}$; $k_i=\frac{b_i}{b}$ (b — standard value of displacement vector); $\sigma_i=\frac{S_i}{p}$; S_i — stress field acting on

the $\perp_i$ when it plays a role of locked dislocation, $\sigma_i^0 = \frac{S_i}{p}$; S_i^0 — stress field acting on $\perp_i$ just after energy release of the $\perp_{i-1}$ dislocation (i—2 — replica).

The relation (10) has two coefficients α, γ which have to be determined from the observational data. The experimental verification of the relations obtained consists thus on the one hand in determining the values α and γ, lying within the permissible limits, and on the other hand with a greater number of replica observations in the conformity of the general form of the relation (10) with the curve (T_i, b_i) for the given series of replicas. Practically, however, the situation looks different. The fargoing simplification of the theory, together with the complexity of the problem, cause wide scattering of the observational values and, in consequence, difficulties in determining accurately the values α and γ. For these reasons we will assume in this analysis $\gamma = 1$ ($p = S$). A further difficulty in computation is caused by the fact that we do not know the full dislocation series but only those dislocations which have become apparent in the earthquake.

Preliminary comparison with the observational data leads to the value of viscosity of an order 10^{12} c.g.s. [5].

A full account of this work is given in [5].

INSTITUTE OF GEOPHYSICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT GEOFIZYKI, PAN)

REFERENCES

[1] S. Droste, R. Teisseyre, Sci. Rep. Tohoku Univ., Ser. 5, Geophys. **11** (1959), No. 1.
[2] — , Bull. Seism. Soc. Am., **51** (1960), 57.
[3] — , Annali di Geofisica, **12** (1959), 179.
[4] R. Teisseyre, Acta Geophys. Polon. **8** (1960), 107.
[5] — , Acta Geophys. Polon. **9** (1961), No 1—2.
[6] A. V. Vvedenskaia, Isv. AN SSSR, (1956), 278.
[7] — , Isv. AN SSSR, (1958), 175.
[8] I. D. Eshelby, F. C. Franck, F. R. N. Nabarro, Phil. Mag. **41** (1951), 351.
[9] B. Gutenberg, Ch. F. Richter, *Seismicity of the Earth*, Princeton Univ. Press., 1949.
[10] M. Bath, Ch. F. Richter, Bull. Sei. Soc. Am. **48** (1958), 133.
[11] H. Honda, Sci. Rep. Tohoku Univ., Ser. 5, Geophys. 9, Supp. 1957.
[12] V. I. Keylis-Borok, DAN SSSR, **71** (1959), No. 6.
[13] F. R. N. Nabarro, Phil. Mag. **122** (1951), 1224.
[14] J. M. Burgers, Proc. Acad. Sci. Amst. **42** (1939), 293.
[15] A. H. Cottrell, *Dislocation and plastic, flow in crystals*, Oxford, Clarendon Press, 1956.
[16] P. Byerly, J. DeNoyer, *Energy in earthquakes as computed from geodetic observations* in Contributions in Geophysics vol. I, 17, Pergamon Press, 1958.
[17] H. F. Reid, Physics of the Earth, **6** (1933), 100.
[18] S. Droste [in preparation].
[19] A. A. Griffith, Phil. Trans. Roy. Soc., A. **221** (1920), 163.
[20] — , First. Inter. Cong. Appl. Mech. p. 55 — Delft: 1924.
[21] A. Nadai, *Theory of flow and fracture of solids*, Mc Graw-Hill Comp., 1950.
[22] I. K. Silverman, Quart. Col. Sch. Min. **52** (1957), No. 2—3.
[23] R. G. Wuerker, Quart. Col. Sch. Min. **54** (1959), 3.

БЮЛЛЕТЕНЬ ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

1961

ТОМ IX Резюме статей ВЫПУСК 5

С. БАЛЬЦЕЖИК, **О НЕКОТОРЫХ КЛАССАХ АБЕЛЕВЫХ ГРУПП** . стр. 327—330

Семейство C абелевых групп называем классом, если выполняются следующие условия:

1. Если группа A изоморфна группе, принадлежащей к C, то $A \in C$.

2. Любая подгруппа и гомеоморфный образ группы, принадлежащей к C, принадлежит к C.

3. Расширение группы, принадлежащей к C, при помощи группы, принадлежащей к C, принадлежит к C.

Класс групп C называем слабо полным, когда $A \otimes B \in C$; $\mathrm{Tor}\,(A, B) \in C$, если только A, $B \in C$.

Класс групп C называем совершенным, если из $A \in C$ вытекает $H_n(\mathrm{A}) \in C$ для любого $n > 0$.

В работе приводится пример класса групп, который не является ни слабо полным, ни совершенным.

Э. СОНСЯДА, **ОТРИЦАТЕЛЬНОЕ РЕШЕНИЕ ПЕРВОЙ ТЕСТОВОЙ ПРОБЛЕМЫ И. КАПЛАНСКОГО ОБ АБЕЛЕВЫХ ГРУППАХ И ОДНОЙ ПРОБЛЕМЫ К. БОРСУКА О ГРУППАХ КОГОМОЛОГИИ** . . . . стр. 331—334

Доказывается следующая теорема:

Существуют абелевые группы без кручения X и Y такие, что

(*i*) *X изоморфна с некоторым прямым слагаемым группы Y и Y изоморфна с некоторым прямым слагаемым группы X.*

(*ii*) *X не изоморфна Y.*

Отсюда в силу одной теоремы Ц. Т. Янга, следует отрицательный ответ на следующую проблему К. Борсука.

Имеют-ли два топологические пространства, изоморфные группы когомологии, если они R-эквивалентные?

Л. КУБИК, **НЕКОТОРЫЕ ЗАМЕЧАНИЯ О КЛАССЕ $\mathcal{L}$ РАСПРЕДЕЛЕНИЙ** . . . стр. 335—336

Обозначим через $\mathcal{G}$ класс всех распределений, для которых функция $G(u)$ в формуле (3) дается формулой (4), или (5), или (6).

Имеем следующую теорему:

Теорема. *Класс $\mathscr{L}$ распределений совпадает с классом композиций конечного числа распределений из класса $\mathscr{G}$ и их пределов (в смысле слабой сходимости).*

Эта теорема аналогична следующей характеризации класса бесконечно делимых распределений ([2], § 17, Теорема 5):

Класс бесконечно делимых распределений совпадает с классом композиций конечного числа распределений Пуассона и их пределов (в смысле слабой сходимости).

М. КРУЛЬ, **СЕПАРАБЕЛЬНЫЕ ГРУППЫ I** . . . . стр. 337—344

Пусть $G = \sum\limits_{t\in T}^{*} R_t$ будет полной прямой суммой групп $R_t (t \in T)$ ранга 1. Множество всех различных типов $\tau(R_t)$ слагаемых $R_t (t \in T)$ обозначаем через $\Omega(G)$. Для всякого $\mathfrak{A} \in \Omega(G)$ через $T_{\mathfrak{A}}$ понимаем подмножество всех тех $t \in T$, для которых $\tau(R_t) = \mathfrak{A}$. Подмножество всех типов, принадлежащих к $\Omega(G)$, которые определяются лишь через характеристики с элементами 0 и ∞, обозначаем через $\Omega_{(0,\infty)}(G)$ и $\Omega_*(G) = \Omega(G) \setminus \Omega_{(0,\infty)}(G)$.

Даются следующие теоремы:

Теорема 1. *Полная прямая сумма* $G = \sum\limits_{t\in T}^{*} R_t = \sum\limits_{\mathfrak{A}\in\Omega(G)}^{*} \sum\limits_{t\in T_{\mathfrak{A}}}^{*} R_t$. *групп ранга 1 есть сепарабельная тогда и только тогда, когда исполнены следующие условия:*

(1) *условие минимальности для типов, принадлежащих к* $\Omega_{(0,\infty)}(G)$;

(2) *всякое подмножество типов* $(\mathfrak{A}_t)_{t\in T_0} \subset \Omega_{(0,\infty)}(G)$ *несравняемых есть конечное;*

(3) *множество* $\Omega_*(G) = \Omega(G) \setminus \Omega_{(0,\infty)}(G)$ *конечное;*

(4) *для всякого* $\mathfrak{A} \in \Omega_*(G)$ *множество* $T_{\mathfrak{A}}$ *конечное.*

Теорема 2. *Полная прямая сумма* $G = \sum\limits_{t\in T}^{*} R_t = \sum\limits_{\mathfrak{A}\in\Omega(G)}^{*} \sum\limits_{t\in T_{\mathfrak{A}}}^{*} R_t$ *сепарабельна тогда и только тогда, когда:*

(1′) *для всякого подмножества* $\Omega' \subset \Omega(G)$ *и всякого выбора характеристик* $\chi'_{\mathfrak{A}} \in \mathfrak{A} \, (\mathfrak{A} \in \Omega')$ *существует конечное подмножество* $\Omega'_0 \subset \Omega'$ *такое, что* $\bigcap\limits_{\mathfrak{A}\in\Omega'} \chi'_{\mathfrak{A}} = \bigcap\limits_{\mathfrak{A}\in\Omega'_0} \chi'_{\mathfrak{A}}$;

(2′) *всякое подмножество несравняемых типов из* $\Omega(G)$ *есть конечное;*

(3′) *для всякого* $\mathfrak{A} \in \Omega^*(G)$ *множество* $T_{\mathfrak{A}}$ *конечное.*

К. БОРСУК, **АБСОЛЮТНЫЙ РЕТРАКТ С БЕСКОНЕЧНЫМ ЧИСЛОМ $\mathcal{R}$-СОСЕДЕЙ** . . . . стр. 345—350

Топологические пространства X и Y являются $\mathcal{R}$-равными, если каждое из них гомеоморфно ретракту остального. Пространства, которые не являются $\mathcal{R}$-равными, называем $\mathcal{R}$-разными.

Если X гомеоморфно ретракту Y, но X и Y являются $\mathcal{R}$-разными, то пишем $X \underset{\mathcal{R}}{<} Y$. Если $X \underset{\mathcal{R}}{<} Y$, но не существует пространство Z такое, что $X \underset{\mathcal{R}}{<} Z \underset{\mathcal{R}}{<} Y$,

то X называем $\mathcal{R}$-соседом для Y с левой строоны, а Y — $\mathcal{R}$-соседом для X с правой стороны.

В работе приводится пример 2-мерного абсолютного (компактного) ретракта, для которого существует бесконечно много $\mathcal{R}$-разных соседей с левой стороны равно как $2^{\aleph_0}$ абсолютных ретрактов $\mathcal{R}$-разных, которые являются соседями для X с правой стороны.

В. ПОГОЖЕЛЬСКИЙ, **СВОЙСТВА ОДНОГО СИНГУЛЯРНОГО ИНТЕГРАЛА В ПРОСТРАНСТВЕ** . . . . стр. 351—356

Автор рассматривает некоторые свойства сингулярного интеграла

$$\Phi(x,u) = \int_{\Omega} F(x-y,y,u)\,dy, \tag{1}$$

в котором функция $F(x,y,u)$ определена формулой (2), функция $K(x',y,u)$ удовлетворяет условиям (3), (3′), (5), (5′).

Автор доказал, что функция (1), определенная в множестве $[x \in \Omega,\ u \in \Omega^*]$, удовлетворяет неравенствам (7) и (7′), следовательно она принадлежит к классу $\mathfrak{H}_a^h$ относительно переменной ψ.

В. ПОГОЖЕЛЬСКИЙ, **ЛИНЕЙНАЯ ЗАДАЧА С РАЗРЫВНЫМИ КАСАТЕЛЬНЫМИ ПРОИЗВОДНЫМИ ДЛЯ ГАРМОНИЧЕСКОЙ ФУНКЦИИ В ПРОСТРАНСТВЕ** стр. 357—362

Задача состоит в определении гармонической функции $u(x)$ в n-мерном эвклидовом пространстве, которая в каждой точке $y \in \Omega$ дисков $D_1, D_2 \ldots, D_p$ отличной от точек поверхностей разрыва $S_\nu^{(1)}$, $S_\nu^{(2)}$, ..., $S_\nu^{(q)}$ удовлетворяет соотношению (2) между краевыми значениями нормальной производной и касательными производными. Коэффициенты $a_\nu^{(j)}(y)$ удовлетворяют условию Гельдера отдельно в каждой области $\Omega_\nu^{(i)} \subset \Omega$, функция $f(y)$ принадлежит к классу $\mathfrak{H}_a^h$, это значит выполняет неравенства (5).

Автор ищет решения задачи в виде потенциала простого слоя (6) и получает сингулярное интегральное уравнение (7) для плотности слоя. Автор решает задачу при помощи известной теоремы Шаудера о неподвижной точке преобразования в пространстве Банаха.

К. МОРЭН, **ПРОСТРАНСТВА H_0^A И ИХ ПРИМЕНЕНИЯ К ОБЩИМ РАЗЛОЖЕНИЯМ ПО СОБСТВЕННЫМ ФУНКЦИЯМ** . . . . стр. 363—367

Автор вводит пространства $H_0^A(\Omega)$, являющиеся натуральным обобщением пространств $H_0^m(\Omega)$, выступающих в теории дифференциальных операторов. Доказываются теоремы о свойстве Гильберта — Шмидта вложений $H_0^A(\Omega)$ в пространство $L^2(\Omega)$.

Опираясь на общую спектральную теорему из [4] доказывается теорема о разложениях по обобщенным собственным функциям, обобщающая и уточняющая теоремы типа Гельфанда — Костюшенки.

А. ЭРЕНФОЙХТ и А. МОСТОВСКИЙ, **КОМПАКТНОЕ ПРОСТРАНСТВО МОДЕЛЕЙ ТЕОРИЙ ПЕРВОГО ПОРЯДКА** . . . . стр. 369—373

Авторы доказывают, что для каждой теории T первого порядка имеется семейство S счетных моделей со следующими свойствами: 1°. S содержит в себе с точностью до изоморфизма каждую счетную модель; 2°. принимая в качестве окрестностей модели множества таких моделей M, в которых определенные элементы $a_1, \ldots, a_n$ удовлетворяют определенной формуле, придаем множеству S структуру компактного топологического пространства.

В. КОЛОДЗЕЙ, **НЕКОТОРЫЕ ОБЩИЕ ЛАКУНАРНЫЕ ТЕОРЕМЫ ОБРАЩЕНИЯ ДЛЯ РЕГУЛЯРНЫХ МЕТОДОВ СУММИРОВАНИЯ ТЕПЛИТЦА** . . . . стр. 375—378

В работе приводятся некоторые результаты, касающиеся существования тауберовых условий лакунарного типа для регулярных методов Теплитца.

Согласно первой из приведенных теорем, этим условиям — даже с так наз. лакунарной функцией — удовлетворяет любой метод, для которого имеется метод не менее сильный, конечнострочный. Затем, приводятся критерии для того, чтобы данный метод бесконечнострочный удовлетворял условиям этой теоремы, чтобы он обладал лакунарными условиями с лакунарной функцией (причем, для него может не существовать метод, не менее сильный, конечнострочный), или же чтобы он вообще не удовлетворял таким условиям.

Настоящая работа не содержит доказательств. Эти доказательства будут опубликованы позднее.

Р. ЭНГЕЛЬКИНГ, **О КОМПАКТИФИКАЦИИ ФРОЙДЕНТАЛЯ** . . . стр. 379—383

В работе дается новое доказательство следующей теоремы Фройденталя - Скляренки:

$\mathfrak{S}$-пространство X обладает компактификацией $\tilde{X}$ такой, что $\operatorname{ind}(\tilde{X}-X) \leqslant 0$, *тогда и только тогда, когда X периферически компактно.*

Р. ТАБЕРСКИЙ, **СУММИРОВАНИЕ С ПРИМЕНЕНИЕМ МНОЖИТЕЛЕЙ КОРОВКИНА** . . . . стр. 385—388

В работе исследуется так наз. (K)-метод суммирования рядов, рассматриваемый в частном случае П. П. Коровкиным [1], [2], [4].

Показано, что (K)-метод сильнее метода Чезаро (С, 1); отсюда следуют теоремы, касающиеся точечной и равномерной (K)-суммируемости некоторых тригонометрических рядов.

Б. МЕЛЬНИК, **О ПРОБЛЕМЕ ПОКАЗАТЕЛЬНОГО ЗАТУХАНИЯ В КВАНТОВОЙ ТЕОРИИ** . . . . стр. 389—393

В работе дается математическая формулировка проблемы показательного затухания в квантовой теории. Дискутируется связь этой формулировки

с задачей асимптотического уравнения для $\psi_{||}$, ($\psi_{||}$ — отображение вектора состояния на установленное подпространство), а также связь с квантовой теорией затухания Круликовского и Жевуского.

ЗИРО КОБА, **ВЕСОВОЙ ФАКТОР МОМЕНТА ИМПУЛЬСА В СТАТИСТИЧЕСКОЙ ТЕОРИИ МНОЖЕСТВЕННОГО ОБРАЗОВАНИЯ ЧАСТИЦ. II** . . . . . . стр. 395—401

Исходя из соотношений данных в первой части получены асимптотические выражения в случае $n \gg 1$ и $\Sigma l_i^2 \gg l_j^2$ для весового фактора момента импульса Z_n в статистической теории множественного образования частиц.

Е. ИГРАС, **ОТОБРАЖЕНИЕ ЭЛЕКТРИЧЕСКОЙ СТРУКТУРЫ ПОВЕРХНОСТЕЙ ПОЛУПРОВОДНИКОВ МЕТОДОМ ЭЛЕКТРОННОГО ЗЕРКАЛА** . . . . стр. 403—407

В работе представлены некоторые результаты наблюдений электрических неоднородностей на поверхностях полупроводниковых кристаллов. Наблюдения проводились при помощи электронного зеркала, дающего увеличение порядка $100\times$.

Исследовались тянутые германиевые $p - n$ переходы, электрические структуры на границах зерен в германии и электрические неоднородности на поверхностях кристаллов кремния. Подчеркивается, что полученные результаты пока временные и исследования поверхностей полупроводников при помощи вышеупомянутого метода в дальнейшем продолжаются.

Л. КОЗЛОВСКИЙ и С. КУБЯК, **ВЛИЯНИЕ КАТОДНОГО ВОДОРОДА НА МАГНИТНЫЕ СВОЙСТВА ЭЛЕКТРОЛИТИЧЕСКИХ ОСАДКОВ НИКЕЛЯ** . .стр. 409—413

Исследовано влияние катодного водорода на магнитные свойства осадков никеля толщиной $7{,}5\mu$ и $14{,}1\mu$, осаждаемых электролитическим путем на медных трубках. Во время насыщения образцов катодным водородом коэрцитивная сила Hc увеличивается, а намагниченность J_{200} (в поле $H = 200$ Oe) и остаточная намагниченность Jr резко уменьшаются.

После шестикратной катодной поляризации осадка никеля, изменения J_{200} и J_s (намагниченность насыщения), в зависимости от объема V_H десорбированного водорода, являются обратимыми.

Констатировано, что катодный водород вызывает два эффекта:

1. Магнетическое отверждение осадков Ni и связанный с этим рост Hc.

2. Образование фазы H/Ni с уменьшенной на 50% намагниченностью насыщения.

Эта фаза с атомной концентрацией 0,6 H/Ni является нестабильной и исчезает после нескольких часов десорбции водорода.

И. ГИНТЕР и В. ШИМАНСКАЯ, **ИЗМЕРЕНИЯ ТЕРМО-ЭДС JnSb** . стр. 419—421

В работе описываются результаты измерений термо-эдс пяти образцов InSb с разными концентрациями примесей в температурном интервале

100—400°K. Во время измерений применялся следующий технический прием: разница температур концов исследуемого образца измерялась термопарами, которые были прикреплены прямо к образцу электролитически нанесенным слоем меди. Концентрация примесей была вычислена на основании измерений коэффициента Холла в температуре жидкого азота.

В работе дается применение теории дислокаций в непрерывных средах к вопросам механизма возникновения землетрясений и условий освобождения внутренней упругой энергии Земли. Полученные при помощи этого метода результаты были сопоставлены с наблюдениями. В частности, объясняется приближенный статистический закон, касающийся количества землетрясений, а также их механизм и величина энергии землетрясени.

Сопоставление размеров дислокаций на поверхности Земли и формул для энергии землетрясений с результатами работы [16] позволило вычислить глубину землетрясений из геодезических данных.

Исследования уравнения движения перемещений, а также вопроса сопротивляемости материала, позволили сформулировать элементарную теорию реплик землетрясений.

TABLE DES MATIÈRES

Cena zł 20.—